Jurnal Ilmiah Matematika dan Pendidikan Matematika (JMP)
Vol. 11 No. 2, Desember 2019, hal. 19-28
ISSN (Cetak) : 2085-1456; ISSN (*Online*) : 2550-0422

THE EXISTENCE AND UNIQUENESS OF THE MILD SOLUTION TO A NONLINEAR CAUCHY PROBLEM ASSOCIATED WITH A NONLOCAL REACTION-DIFFUSION SYSTEM

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ABSTRACT. *We study the existence and uniqueness of a mild solution to a nonlinear Cauchy problem associated with a nonlocal reaction diffusion system by employing the properties of analytic semigroup operator generated by the linear part of the problem which is sectorial and then applying Banach Fixed Point Theorem to the problem. We show that the problem has a unique mild solution under a Lipschitz condition on the nonlinear part of the problem. An example as an application of the result obtained is also given.*
Keywords : *existence, uniqueness, mild solution, nonlinear Cauchy problem, Banach fixed point theorem.*

ABSTRAK. Kita mengkaji keujudan dan ketunggalan penyelesaian lemah masalah Cauchy nonlinier yang berkaitan dengan sistem reaksi difusi nonlokal dengan menggunakan sifat-sifat operator semigrup yang dibangkitkan oleh operator pada bagian liniernya yang bersifat sektorial dan kemudian menerapkan Teorema Titik Tetap Banach pada masalah tersebut. Kita tunjukkan bahwa masalah tersebut memiliki penyelesaian lemah yang unik atas kondisi Lipschitz untuk operator pada bagian nonliniernya. Satu contoh sebagai penerapan hasil yang diperoleh juga diberikan.
Kata kunci : keujudan, ketunggalan, penyelesaian lemah, masalah Cauchy nonlinier, teorema titik tetap Banach.

1. INTRODUCTION

Consider the system

$$\frac{\partial q_A}{\partial t} = D_A \Delta q_A - k_A \int_{B(\cdot,R)\cap\Omega} q_B dy \cdot q_A, \quad \text{in } \Omega \times (0,T),$$

$$\frac{\partial q_B}{\partial t} = D_B \Delta q_B - k_B \int_{B(\cdot,R)\cap\Omega} q_A dy \cdot q_B, \quad \text{in } \Omega \times (0,T),$$

$$\frac{\partial q_A}{\partial n} = \frac{\partial q_B}{\partial n} = 0, \quad \text{on } \partial\Omega \times (0,T), \tag{1.1}$$

with $\Omega \subset \mathbb{R}^n$ is a bounded domain with C^2 boundary. The system describes the chemical reaction

$$A + B \longrightarrow C$$

that occurs if and only if a pair $A - B$ molecules are in a distance less than R. A molecule of A reacts with a molecule of B to produce a molecule of C (see [1]).

In this paper, we study the existence and uniqueness of the mild solution to a Cauchy problem associated with the problem (1.1)

$$D_t u(t) = Au(t) + f\big(u(t)\big), \qquad 0 < t \leq T$$

$$u(0) = u_0 \tag{1.2}$$

where X is a Banach space, D_t is derivative of first order with respect to t, $A : D(A) \subseteq X \to X$ is a sectorial linear operator, $u_0 \in X$, and $f : X \to X$ is a nonlinear mapping satisfying the lipschitz condition, that is there exists $K > 0$ such that

$$\|f(u) - f(v)\| \leq K\|u - v\|, \;\; u, v \in X.$$

We employ the properties of analytic semigroup generated by the operator A of the problem (1.2) and then apply Banach Fixed Point Theorem to show the existence and uniqueness of a mild solution to the problem (1.2) under certain conditions.

This paper is composed of four sections. In the second section, we provide briefly some properties of analytic semigroup. In the third section, we prove the existence and uniqueness of the mild solution to the problem (1.2). In the last section, we give an example as an application of the result obtained.

2. PRELIMINARIES

In this section, we provide briefly some results concerning solution operators for the Cauchy problem

$$D_t u(t) = Au(t) + f(t), \qquad t > 0$$
$$u(0) = u_0 \tag{2.1}$$

DEFINITION 2.1. *An operator A is called sectorial if A satisfies the properties that there are constant $\theta \in \left(\frac{\pi}{2}, \pi\right)$ and $M > 0$ such that $\rho(A) \supset S_\theta = \{\lambda \in \mathbb{C}: \lambda \neq 0, |arg(\lambda)| < \theta\}$,*

$$\|R(\lambda, A)\| \leq \frac{M}{|\lambda|}, \qquad \lambda \in S_\theta,$$

where $R(\lambda, A) = (\lambda I - A)^{-1}$ and $\rho(A) = \{\lambda \subseteq \mathbb{C} \mid R(\lambda, A) \text{ is bounded}\}$ are the resolvent operator and resolvent set of A, respectively.

DEFINITION 2.2. *For $r > 0$ and $\frac{\pi}{2} < \omega < \theta$, $\Gamma_{r,\omega} := \{\lambda \in \mathbb{C} : |\arg(\lambda)| = \omega, |\lambda| \geq r\} \cup \{\lambda \in \mathbb{C} : |\arg(\lambda)| \leq \omega, |\lambda| = r\}$*

The linear operator A generates a solution operator for the problem (2.1), that is

$$S(t) = \frac{1}{2\pi i} \int_{\Gamma_{r,\omega}} e^{\lambda t} R(\lambda; A)\, d\lambda, \qquad t > 0, \tag{2.2}$$

where $\Gamma_{r,\omega}$ is counterclockwise. By Cauchy's theorem, the integral form (2.2) is independent of $r > 0$ and $\omega \in (\pi/2, \theta)$.

Let $B(X)$ be the set of all bounded linear operators on X. The properties of the family $\{S(t)\}_{t>0}$ are given in the following theorems.

THEOREM 2.3. *Let A be a sectorial operator and $S(t)$ be the operator defined by (2.2). Then the following statements hold.*

(i) $S(t) \in B(X)$ *and there exists* $C_1 > 0$ *such that*

$$\|S(t)\| \leq C_1, \qquad t > 0.$$

(ii) $S(t) \in B\big(X\colon D(A)\big)$ *for* $t > 0$ *and if* $x \in D(A)$ *then* $A\,S(t)x = S(t)Ax$. *Moreover, there exists* $C_2 > 0$ *such that*

$$\|AS(t)\| \le C_2 t^{-1}, \qquad t > 0,$$

where $B\big(X\colon D(A)\big) = \{T\colon X \to D(A) \mid T \text{ is bounded operator}\}$ *and* $D(A) \subseteq X$.

(iii) *The function* $t \mapsto S(t)$ *belongs to* $C^{\infty}\big((0,\infty); B(H)\big)$ *and it holds that*

$$S^{(n)}(t) = \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} e^{t\lambda}\lambda^n R(\lambda; A)\ d\lambda, n = 1,2,3, \ldots$$

and there exists $M_n > 0, n = 1,2,3, \ldots$ *, such that*

$$\left\|S^{(n)}(t)\right\| \le M_n t^{-n}, \qquad t > 0.$$

Moreover, it has analytic continuation $S(z)$ *to the sector* $S_{\theta-\frac{\pi}{2}}$ *and for* $z \in S_{\theta-\frac{\pi}{2}}, \eta \in \left(\frac{\pi}{2}, \theta\right)$, *it holds that*

$$S(z) = \frac{1}{2\pi i}\int_{\Gamma_{r,\eta}} e^{\lambda z} R(\lambda; A)\ d\lambda.$$

(iv) *For* $x \in X$,

$$S(t)S(s) = S(t+s), \qquad s,t > 0.$$

THEOREM 2.4. *Let A be a sectorial operator and*

$$S(t) = \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} e^{\lambda t} R(\lambda; A)\ d\lambda.$$

Then the following statements hold.

(i) *If* $x \in \overline{D(A)}$ *then* $lim_{t\to 0^+} S(t)x = x$.

(ii) *For every* $x \in D(A)$ *and* $t > 0$,

$$\int_0^t S(\tau)x\,d\tau \in D(A),$$

$$\int_0^t AS(\tau)x\,d\tau = S(t)x - x.$$

(iii) *If* x, $Ax \in D(A)$ *then*

$$\lim_{t\to 0^+}\frac{S(t)x - x}{t} = Ax.$$

For more discussion concerning analytic semigroup operator and its properties, one can refer to [2,3].

3. MAIN RESULTS

In this section, we study the existence and uniqueness of the solution to the Cauchy problem (1.2)

$$D_t u(t) = Au(t) + f\big(u(t)\big), \qquad 0 < t \le T,$$
$$u(0) = u_0$$

where A is a sectorial linear operator, $f: X \to X$, and $u_0 \in X$. We start defining the mild solution for this problem.

DEFINITION 3.1. *A function* $u: [0, T] \to X$ *is a mild solution to the problem* (1.2) *if it satisfies*

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f\big(u(s)\big)ds, \qquad 0 < t \le T. \tag{3.1}$$

The following theorem shows us the existence and uniqueness of the mild solution to the Problem (3.1) by assuming the Lipschitz condition on f.

THEOREM 3.2. *Let* $f: X \to X$ *and there exists* $K > 0$ *such that*

$$\|f(u) - f(v)\| \le K\|u - v\|, \;\; u, v \in X.$$

If $u_0 \in \overline{D(A)}$ *then there exists* $T_0 > 0$ *such that the problem (2.1) has a unique mild solution* $u \in C([0, T_0]; X)$.

Proof. Given $\varepsilon > 0$. Since $u_0 \in \overline{D(A)}$, we have $\|S(t)u_0 - u_0\| \to 0$ as $t \to 0^+$ by Theorem 2.4(i). It means that there exists $0 < \tau \le T$, such that

$$\|S(t)u_0 - u_0\| \le \frac{\varepsilon}{2}, \qquad 0 < \tau \le T.$$

We next suppose the Banach space $Z = C([0, T_0]; X)$ and its subset $Y = \{u \in Z: \|u - u_0\|_Z \le \varepsilon)$ with

$$T_0 = \min\left\{\tau, \frac{\varepsilon}{2C(K\varepsilon + \|f(\,u_0)\|)}\right\}.$$

We now define a mapping F on Y by

$$F(u_0) := u_0; Fu(t) := S(t)u_0 + \int_0^t S(t-s)f\big(u(s)\big)\,ds, \qquad 0 < t \le T_0.$$

We prove that $Fu \in Y$ or, in other words, $Fu \in Z$ and $\|Fu - u_0\| \le \varepsilon$. Note that

$$\begin{aligned}\|Fu(t) - u_0\| &= \|Fu(t) - S(t)u_0 + S(t)u_0 - u_0\| \\ &\le \|Fu(t) - S(t)u_0\| + \|S(t)u_0 - u_0\| \\ &= \left\|\int_0^t S(t-s)f\big(u(s)\big)\,ds\right\| + \|S(t)u_0 - u_0\| \\ &\le \int_0^t \|S(t-s)\|\,\big\|f\big(u(s)\big)\big\|ds + \|S(t)u_0 - u_0\|.\end{aligned}$$

By Theorem 2.3(i), we obtain

$$\|Fu(t) - u_0\| \le C\int_0^t \big\|f\big(u(s)\big)\big\|\,ds + \|S(t)u_0 - u_0\|.$$

By using Lipschitz condition, and since $u \in Y$, for $0 < s < t \le T_0 < \tau$, we obtain

$$\big\|f\big(u(s)\big)\big\| \le K\varepsilon + \|f(u_0)\|. \tag{3.2}$$

Therefore

$$\|Fu(t) - u_0\| \le C(K\varepsilon + \|f(u_0)\|)T_0 + \frac{\varepsilon}{2}.$$

implying

$$\|Fu - u_0\|_Z \le C(K\varepsilon + \|f(u_0)\|)T_0 + \frac{\varepsilon}{2} < \varepsilon.$$

We now prove that $Fu \in Z$ or continuity of Fu in $t \in [0, T_0]$. Consider that, for $0 < t \le T_0$,

$$\begin{aligned}&\|Fu(t+h) - Fu(t)\| \\ &\quad\le \|S(t+h)u_0 - S(t)u_0\| \\ &\qquad + \left\|\int_0^{t+h} S(t+h-s)f\big(u(s)\big)\,ds - \int_0^t S(t-s)f\big(u(s)\big)\,ds\right\| \\ &\quad\le \|S(t+h)u_0 - S(t)u_0\| + \int_0^t \|S(t-s)\|\,\big\|f\big(u(s+h)\big) - f\big(u(s)\big)\big\|ds \\ &\qquad + \int_0^h \big\|S(t+h-s)f\big(u(s)\big)\big\|\,ds\end{aligned}$$

$$\leq \|S(h)S(t)u_0 - S(t)u_0\| + \int_0^t \|S(t-s)\| \left\|f\big(u(s+h)\big) - f\big(u(s)\big)\right\| ds$$
$$+C(K\varepsilon + \|f(u_0)\|)h$$

By (3.2), the Lipschitz condition on f, and the Dominated Convergence Theorem,

$$\int_0^t \|S(t-s)\| \left\|f\big(u(s+h)\big) - f\big(u(s)\big)\right\| ds, \text{ as } h \to 0.$$

Therefore $\|Fu(t+h) - Fu(t)\| \to 0$ as $h \to 0$ for $0 < t \leq T_0$. For $t = 0$, we have

$$\|Fu(h) - Fu(0)\| = \|Fu(h) - u_0\|$$
$$\leq \|S(h)u_0 - u_0\| + \int_0^h \left\|S(h-s)f\big(u(s)\big)\right\| ds$$
$$\leq \|S(h)u_0 - u_0\| + C(K\varepsilon + \|f(u_0)\|)h.$$

Then $\|Fu(t+h) - Fu(t)\| \to 0$ as $h \to 0$ for $t = 0$. Consequently, Fu continuous in $t \in [0, T_0]$ or $Fu \in Z$. Thus $F: Y \to Y$.

Next, we prove that $F: Y \to Y$ is a contraction. Observe that

$$\|Fu(t) - Fv(t)\|$$
$$= \left\| S(t)u_0 + \int_0^t S(t-s)f\big(u(s)\big)\, ds - S(t)u_0 - \int_0^t S(t-s)f\big(v(s)\big)\, ds \right\|$$
$$\leq \int_0^t \|S(t-s)\| \left\|f\big(u(s)\big) - f\big(v(s)\big)\right\| ds$$
$$\leq CK \int_0^t \|u(s) - v(s)\|\, ds.$$

Since $u \in Y \subseteq Z = \{u: [0, T_0] \to X : u \text{ is continuous on } [0, T_0]\}$,

$$\|Fu(t) - Fv(t)\| \leq T_0 CK \|u - v\|_Z.$$

This implies

$$\|Fu - Fv\|_Z \leq T_0 CK \|u - v\|_Z .$$

Since

$$T_0 CK\varepsilon < T_0 C \left(K\varepsilon + \|f(\, u_0)\|_{L^\infty((0,T];X)} \right) \leq \frac{\varepsilon}{2},$$

we have $T_0CK < 1/2$. Therefore $F: Y \to Y$ is a contraction. By Banach Fixed Point Theorem, F has a unique fixed point $u \in Y$. Here we have

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f\big(u(s)\big)ds, \qquad 0 < t \le T.$$

Note that since $u_0 \in \overline{D(A)}$, then by 3.2, Theorem 2.3(i), and Theorem 4.12(i), we obtain $\|u(t) - u_0\| \to 0$ as $t \to 0^+$. It follows that $u \in C([0, T_0]; X)$.

4. APPLICATIONS

Consider again the system (1.1)

$$\frac{\partial q_A}{\partial t} = D_A \Delta q_A - k_A \int_{B(\cdot,R)\cap\Omega} q_B dy \cdot q_A, \ \text{ in } \Omega \times (0,T),$$

$$\frac{\partial q_B}{\partial t} = D_B \Delta q_B - k_B \int_{B(\cdot,R)\cap\Omega} q_A dy \cdot q_B, \ \text{ in } \Omega \times (0,T),$$

$$\frac{\partial q_A}{\partial n} = \frac{\partial q_B}{\partial n} = 0, \ \text{ on } \partial\Omega \times (0,T),$$

with $\Omega \subset \mathbb{R}^n$ is a bounded domain with C^2 boundary. The abstract formulation of the problem (1.1) is

$$D_t U = AU + F(U), \ \ 0 < t \le T,$$
$$U(0) = U_0$$

in

$$X = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} : u, v \in L^2(\Omega) \right\}$$

where

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, U_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, D_t U = \begin{pmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial u}{\partial t} \end{pmatrix}, A = \begin{bmatrix} D_A \Delta & 0 \\ 0 & \Delta D_B \end{bmatrix}, \text{and } F = \begin{pmatrix} f(u,v) \\ f(v,u) \end{pmatrix}$$

with

$$f(u,v) = u\bar{v} \text{ and } \bar{v} = \int_\Omega \chi(\cdot, y) v(y) dy,$$

where $\chi(x,y) = \chi_{B(x,R)\cap\Omega}(y)$ denoting the characteristic function of $B(x,R)\cap\Omega$. We next set

$$D(A) = \left\{\begin{pmatrix} u \\ v \end{pmatrix} : u, v \in H_N^2(\Omega)\right\}$$

where

$$H_N^2(\Omega) = \left\{u \in H^2(\Omega) : \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\right\}.$$

The operator Δ is dissipative and self adjoint implying that Δ is sectorial in X. Moreover, for any $\lambda \in S_\theta$ with $\theta \in (\pi/2, \pi)$, we have

$$(\lambda - A)^{-1} = \begin{bmatrix} (\lambda - D_A\Delta)^{-1} & 0 \\ 0 & (\lambda - D_B\Delta)^{-1} \end{bmatrix}.$$

Thus, there exists $M > 0$ such that $\|(\lambda - A)^{-1}\| \leq M/|\lambda|$ for all $\lambda \in S_\theta$.

Note that since

$$\|\bar{u}\|_2^2 \leq |\Omega|^4 \|u\|_2^2,$$

then, for

$$U = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, V = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \in D(A),$$

we get

$$\begin{aligned}
&\|F(U) - F(V)\|_2^2 \\
&\quad \leq \|u_1\bar{v}_1 - u_2\bar{v}_2\|_2^2 + \|v_1\bar{u}_1 - v_2\bar{u}_2\|_2^2 \\
&\quad = \|(u_1 - u_2)\bar{v}_1 + u_2(\bar{v}_1 - \bar{v}_2)\|_2^2 + \|(v_1 - v_2)\bar{u}_1 + v_2(\bar{u}_1 - \bar{u}_2)\|_2^2 \\
&\quad \leq (\|u_1 - u_2\|_2\|\bar{v}_1\|_2 + \|u_2\|_2\|\bar{v}_1 - \bar{v}_2\|_2)^2 \\
&\qquad +(\|v_1 - v_2\|_2\|\bar{u}_1\|_2 + \|v_2\|_2\|\bar{u}_1 - \bar{u}_2\|_2)^2 \\
&\quad \leq |\Omega|^4(\|u_1 - u_2\|_2\|v_1\|_2 + \|u_2\|_2\|v_1 - v_2\|_2)^2 \\
&\qquad +(\|v_1 - v_2\|_2\|u_1\|_2 + \|v_2\|_2\|u_1 - u_2\|_2)^2 \\
&\quad \leq |\Omega|^4(\|u_1 - u_2\|_2 + \|v_1 - v_2\|_2)^2 \\
&\qquad +((\|v_1\|_2 + \|u_2\|_2)^2 + (\|u_1\|_2 + \|v_2\|_2)^2) \\
&\quad \leq 4|\Omega|^4(\|u_1 - u_2\|_2^2 + \|v_1 - v_2\|_2^2) \\
&\qquad +(\|u_1\|_2^2 + \|u_2\|_2^2 + \|v_1\|_2^2 + \|v_2\|_2^2) \\
&\quad = 4|\Omega|^4(\|U\|_2^2 + \|V\|_2^2)\|U - V\|_2^2.
\end{aligned}$$

Consequently, since $\|(u,v)\|_\infty \leq \|(u_0, v_0)\|_\infty$, there exists $K > 0$ such that

$$\|F(U) - F(V)\|_2 \leq K\|U - V\|_2.$$

Thus, since A is a sectorial operator, based on Theorem 3.2, if $U_0 \in \overline{D(A)}$ then the problem (1.1) has a unique mild solution $U \in C([0, T_0]; X)$ for some $T_0 > 0$.

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