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A NEW APPROACH TO BOUNDING THE PERIMETER OF AN ELLIPSE: EXTREMAL ANALYSIS AND INTEGRAL REFORMULATION

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ABSTRACT: *The perimeter of an ellipse is typically represented using an elliptic integral, which does not have a closed-form solution. One common approach to approximating the ellipse perimeter involves determining its bounds using arithmetic, geometric, and harmonic means. This study refines the integral formulation representing the ellipse perimeter as the average of its integral. By restructuring the integration limits, a new method is introduced to determine the bounds based on the extreme values of the integrand. This approach offers a clearer geometric interpretation, establishes a new lower bound for the perimeter, and proposes a conjecture for the greatest possible lower bound, supported by analysis of the trapezoidal rule with a single subinterval.*

Keywords: *Ellipse perimeter, elliptic integral, perimeter bounds, integral reformulation, conjecture.*

ABSTRAK: Keliling elips umumnya dinyatakan dalam bentuk integral eliptik, yang tidak memiliki bentuk tertutup. Salah satu pendekatan yang umum digunakan untuk mendekati nilai keliling elips adalah dengan menentukan batas-batasnya menggunakan rata-rata aritmetika, geometri, dan harmonik. Studi ini menyempurnakan formulasi integral yang merepresentasikan keliling elips sebagai rata-rata dari suatu integral. Dengan merestrukturisasi batas-batas integrasi, diperkenalkan sebuah metode baru untuk menentukan batas-batas tersebut berdasarkan nilai ekstrem dari fungsi integran. Pendekatan ini memberikan interpretasi geometris yang lebih jelas, menetapkan batas bawah baru untuk keliling, dan mengusulkan sebuah dugaan untuk batas bawah terbesar yang mungkin, yang didukung oleh analisis aturan trapesium dengan satu subinterval.

Kata Kunci: Keliling elips, integral eliptik, batas keliling, reformulasi integral, dugaan (konjektur).

1. INTRODUCTION

Several mathematicians have developed methods to approximate the ellipse perimeter by establishing its bounds. This pursuit is crucial since the ellipse perimeter is represented as an elliptic integral with no known analytical solution. Wang et al. (2014) refined the ellipse perimeter bounds that are typically obtained using arithmetic, geometric, and harmonic means (AGHM) (M. K. Wang et al., 2014). Other studies have explored circle–ellipse relationships to estimate the

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ellipse perimeter, though this is more challenging than relating their areas (Rohman, 2022, 2024).

An intriguing study was conducted by Klamkin, Pain, Jameson, Gusić, and Pfiefer, who employed the following elliptic integral:

$$L(a,b) = 4\int_0^{\pi/2} \frac{1}{2}\left(\sqrt{a^2\cos^2(t) + b^2\sin^2(t)} + \sqrt{a^2\sin^2(t) + b^2\cos^2(t)}\right) dt \tag{1}$$

for $a \geq b \geq 0$. Pain, Jameson, and Gusić derived equation (1) by utilizing the symmetry properties of elliptical functions, whereas Pfiefer employed curve-length inequalities based on the Minkowski Sums concept (Gusić, 2015; Jameson, 2014; Klamkin, 1971; Pain, 2022; Pfiefer, 1988).

In their search for perimeter bounds, Pain (2022), Gusić (2015), and Jameson (2014) applied a combination of AGHM, while Pfiefer leveraged the Cauchy-Schwarz inequality. They established the following perimeter bounds:

$$2\pi\frac{a+b}{2} \leq L(a,b) \leq 2\pi\sqrt{\frac{a^2+b^2}{2}}$$

Meanwhile, Pain found that $2\pi\sqrt{ab} \leq L(a,b)$, which can actually be refined to produce the same lower bound. When $a = b$ the expression for $L(a,b)$ reduces to the perimeter of a circle.

While previous studies have focused on the complete elliptic integral of the second kind to derive the ellipse perimeter from equation (1), namely:

$$L(a,b) = 4a.E(e)$$

where $E(e)$ denotes the complete elliptic integral of the second kind and the eccentricity is given by $e = \sqrt{1 - b^2/a^2}$, with $a \geq b$, as well as other techniques such as the AGHM, this study introduces a fundamentally different approach (Chandrupatlla & Osler, 2010; Liang, 2019).

Although various bounding techniques have been extensively developed, they still face a significant limitation. Most existing methods offer limited geometric interpretation and rely heavily on algebraic manipulation rather than structural analysis of the integral itself. This shortcoming highlights the need for developing new approaches that directly analyze the behavior of the integrand in equation (1), aiming to produce sharper bounds and provide deeper geometric insight.

By analyzing the extreme values of the integrand in equation (1) and applying the squeeze theorem, this work refines the integration limits—a modification not attempted in earlier studies. This novel strategy not only simplifies the derivation process but also leads to new, tighter bounds for the ellipse perimeter.

Another contribution of this study is the dual geometric interpretation of the refined bounds: one based on the area under the curve (Figure 2), and the other on the mean distance between the ellipse's radii (Figures 1 and 3). This study also improves upon Pfiefer's original illustration (Figure 1) by providing a clearer visualization in Figure 3, which more accurately represents the geometric behavior of the integrand. These interpretations offer deeper insights beyond traditional AGHM methods.

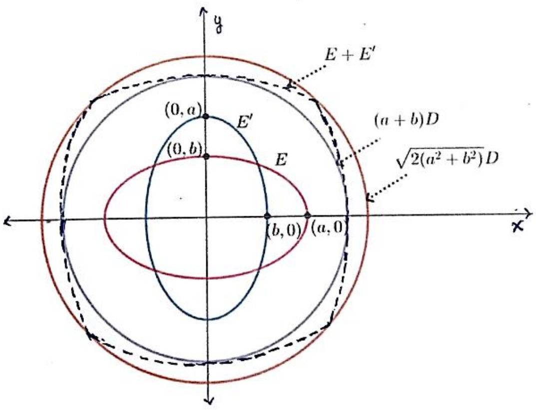


Figure 1. Illustration of Ellipse Perimeter Bounds from Pfiefer (1988). Pfiefer depicted the dashed curve with perimeter $\boldsymbol{2L(a,b)=E+E'}$, where $\boldsymbol{E=E'}$ (Pfiefer, 1988).

In summary, the key contributions of this study are twofold: the introduction of a refined integration scheme using extremal values of the integrand, and the development of two distinct geometric interpretations for the resulting perimeter bounds. These innovations offer significant improvements over the existing methods, delivering both a theoretical and practical advance in the approximation of the ellipse perimeter.

2. RESULTS AND DISCUSSION

2.1 Results

We can define equation (1) as the average integral of the ellipse perimeter:

$$L(a,b) = \frac{1}{2}\left[4\int_0^{\pi/2} \sqrt{a^2\cos^2(t) + b^2\sin^2(t)}\ dt + 4\int_0^{\pi/2} \sqrt{a^2\cos^2(t) + b^2\sin^2(t)}\ dt\right]$$

which simplifies to:

$$L(a,b) = \frac{1}{2}[L_1(a,b) + L_2(a,b)] \tag{2}$$

where $L_1(a,b)$ denotes the perimeter of an ellipse parameterized by $(b\cos(t), a\sin(t))$, and $L_2(a,b)$ corresponds to an ellipse given by $(a\cos(t), b\sin(t))$. Figure 1 illustrates this relationship, where $L_1(a,b) = E'$ and $L_2(a,b) = E$.

2.1.1 Reformulation of Equation (1) with a Shorter Interval

The refinement of equation (1) involves shortening the integration interval to $[0, \pi/4]$ while adjusting the multiplication factor to 8. This adjustment not only simplifies the analysis but also enhances the geometric interpretability of the upper and lower bounds, as illustrated in Figure 2. From a geometric perspective, when the integrand of equation (1) is plotted, it is bounded below by $(a+b)/2$ and above by $\sqrt{(a^2+b^2)/2}$ These bounds are precisely realized when the limits of integration are set to 0 and $\pi/4$.

This refinement is formalized in the following theorem:

Theorem 2.1. The perimeter of ellipse $L(a,b)$ in equation (1) is equivalent to:

$$L(a,b) = 8\int_0^{\pi/4} \frac{1}{2}\left(\sqrt{a^2\cos^2(t) + b^2\sin^2(t)} + \sqrt{a^2\sin^2(t) + b^2\cos^2(t)}\right)\ dt$$

for $a \geq b \geq 0$.

Proof. The perimeter of the ellipse can be expressed as equation (1)

$$L(a,b) = 8\int_0^{\pi/2} \frac{1}{2}\left(\sqrt{a^2\cos^2(t) + b^2\sin^2(t)} + \sqrt{a^2\sin^2(t) + b^2\cos^2(t)}\right)\ dt.$$

Let

$$f(t) = \frac{1}{2}\left(\sqrt{a^2\cos^2(t) + b^2\sin^2(t)} + \sqrt{a^2\sin^2(t) + b^2\cos^2(t)}\right).$$

Define

$$I_1 = \int_0^{\pi/4} \sqrt{a^2\cos^2(t) + b^2\sin^2(t)}\,dt, \quad I_2 = \int_{\pi/4}^{\pi/2} \sqrt{a^2\sin^2(t) + b^2\cos^2(t)}\,dt.$$

Then,

$$L(a,b) = 4(I_1 + I_2). \tag{3}$$

Since $f(t)$ is symmetric with respect to $\pi/2$, we can verify that $f(t) = f\left(\frac{\pi}{2} - t\right)$.

To establish equivalence with equation (1), consider I_2. By making the substitution $u = \pi/2 - t$ gives $du = -dt$, which modifies the interval as follows:

- when $t = \frac{\pi}{4}$, then $u = \pi/4$
- when $t = \frac{\pi}{2}$, then $u = 0$

Applying trigonometric identities:

$$\sin^2(t) = \cos^2\left(\frac{\pi}{2} - u\right) = \cos^2(u)\,, \text{and} \;\; \cos^2(t) = \sin^2\left(\frac{\pi}{2} - u\right) = \sin^2(u).$$

Thus,

$$I_2 = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{a^2\cos^2(t) + b^2\sin^2(t)}\,dt = \int_{\frac{\pi}{4}}^{0} \sqrt{a^2\cos^2(u) + b^2\sin^2(u)}\,(-du).$$

Reversing the integration limits removes the negative sign:

$$I_2 = \int_0^{\pi/4} \sqrt{a^2\cos^2(u) + b^2\sin^2(u)}\,du.$$

Replacing the dummy variable back u with t does not alter the result, so:

$$I_2 = \int_0^{\pi/4} \sqrt{a^2\cos^2(t) + b^2\sin^2(t)}\,dt. \tag{4}$$

Substituting equation (4) into equation (3) yields:

$$4(I_1 + I_2) = 4\left(\int_0^{\frac{\pi}{4}} \sqrt{a^2\cos^2(t) + b^2\sin^2(t)}\,dt \right.$$
$$\left. + \int_0^{\frac{\pi}{4}} \sqrt{a^2\cos^2(t) + b^2\sin^2(t)}\,dt\right)$$

which simplifies to:

$$4(I_1 + I_2) = 4\int_0^{\frac{\pi}{4}} \sqrt{a^2\cos^2(t) + b^2\sin^2(t)}\,dt.$$

Since

$$4(I_1 + I_2) = L(a,b).$$

we obtain

$$L(a,b) = 8\int_0^{\pi/4} \frac{1}{2}\left(\sqrt{a^2\cos^2(t) + b^2\sin^2(t)} + \sqrt{a^2\sin^2(t) + b^2\cos^2(t)}\right)dt.$$

Thus, the theorem is proven. ■

2.1.2 Ellipse Perimeter Bounds Using Extreme Values and the Squeeze Theorem

Next, we will prove the ellipse perimeter bound theorem by utilizing the extreme values of the function and the integral squeeze theorem. We prove Theorem 2.2 using Theorem 2.1, which provides a more straightforward approach than the previous proof that relied on the Cauchy–Schwarz and AGHM inequalities, as presented in Jameson (2014), Liang (2019), and M. K. Wang et al. (2014).

Theorem 2.2. The perimeter $L(a,b)$ of an ellipse satisfies the following inequality:

$$2\pi\frac{a+b}{2} \leq L(a,b) \leq 2\pi\sqrt{\frac{a^2+b^2}{2}}$$

for $a \geq b \geq 0$.

Proof. Given that

$$L(a,b) = 8\int_0^{\pi/4} \frac{1}{2}\left(\sqrt{a^2\cos^2(t) + b^2\sin^2(t)} + \sqrt{a^2\sin^2(t) + b^2\cos^2(t)}\right)dt.$$

Define

$$f(t) = \frac{1}{2}\sqrt{a^2\cos^2(t) + b^2\sin^2(t)} + \sqrt{a^2\sin^2(t) + b^2\cos^2(t)}.$$

To find the extrema of $f(t)$, we solve for t such that $f'(t) = 0$. The first derivative is

$$f'(t) = \frac{(b^2 \ - a^2)\sin(2t)}{4}$$
$$\times\left(\frac{1}{\sqrt{b^2\sin^2(t) + a^2\cos^2(t)}} - \frac{1}{\sqrt{a^2\sin^2(t) + b^2\cos^2(t)}}\right).$$

The derivative equals zero when:

- $\sin(2t) = 0$, occurring at $t = k\pi/2$ for $k \in \mathbb{Z}$
- The term inside parentheses is zero:

$$\frac{1}{\sqrt{b^2\sin^2(t) + a^2\cos^2(t)}} - \frac{1}{\sqrt{a^2\sin^2(t) + b^2\cos^2(t)}} = 0$$

  which occurs at $t = \pi/4 + k\,\pi/2$ for $k \in \mathbb{Z}$.

Substituting these values into $f(t)$ gives

$$\min_{t\epsilon\left[0,\frac{\pi}{4}\right]}\{f(t)\} = \frac{a+b}{2}, \qquad \max_{t\epsilon\left[0,\frac{\pi}{4}\right]}\{f(t)\} = \sqrt{\frac{a^2+b^2}{2}}.$$

Using the squeeze theorem, we derive the bounds for $L(a,b)$, which leads to

$$8\int_0^{\pi/4}\frac{a+b}{2}\ dt \le 8\int_0^{\frac{\pi}{4}} f(t)\ dt \le 8\int_0^{\pi/4}\sqrt{\frac{a^2+b^2}{2}}\ dt$$

producing

$$2\pi\frac{a+b}{2} \le 8\int_0^{\frac{\pi}{4}} f(t)\ dt \le 2\pi\sqrt{\frac{a^2+b^2}{2}}. \qquad \blacksquare$$

2.1.3 Visualization of Ellipse Perimeter Bounds

The following figure illustrates the curve $f(t)$ along with its lower bound $(a+b)/2$ and upper bound $\sqrt{(a^2+b^2)/2}$. The integral area $8\int_0^{\frac{\pi}{4}} f(t)\ dt$ is located between $\pi(a+b)$ and $2\pi\sqrt{(a^2+b^2)/2}$.

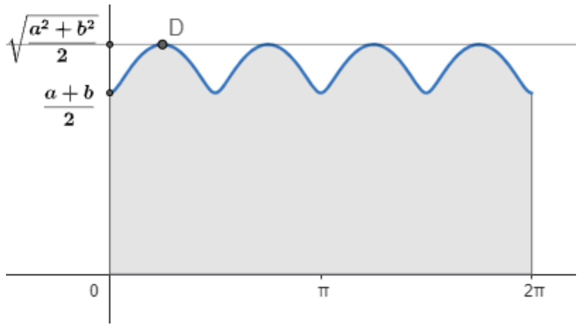


Figure 2. Bounds of the ellipse perimeter $\boldsymbol{L(a,b)}$

2.1.4 Visualization of the Curve for the Mean Integral Function

To present $f(t)$ as a closed curve, we can draw an analogy with the perimeter of a circle:

$$L(r) = \int_0^{2\pi} \sqrt{r^2 \sin^2(t) + r^2 \cos^2(t)}\, dt = \int_0^{2\pi} r\, dt.$$

This equation utilizes the radius function of a circle. By a similar principle, $f(t)$ serves as a radius function for a closed curve with perimeter:

$$L(a,b) = 8\int_0^{\frac{\pi}{4}} f(t)\ dt.$$

To visualize this concept geometrically, consider two points:

- let point A lie on an ellipse $\varepsilon_1 : \frac{x^2}{a^2} + \frac{x^2}{b^2} = 1$,
- let point B lie on another ellipse $\varepsilon_2 : \frac{x^2}{b^2} + \frac{x^2}{a^2} = 1$.

Let O be the origin (center of the ellipses). Denote:

- OA and OB as the length from O to points A and B, respectively;
- Define a new point C, lying on the line segment connecting O and B such that the length OC is defined as the average of OA and OB

$$OC = \frac{1}{2}(OA + OB).$$

Since point C lies along the line extending from OA, we can write

$$OC = k.OA$$

where the scaling factor k is given by

$$k = \frac{OC}{OA} = \frac{OA + OB}{2\,OA}.$$

The coordinates of point *C* can be expressed parametrically as

$$C = O + k(A - O).$$

Visualizing Point *C* Using GeoGebra

To visualize point *C* in a coordinate system, consider an ellipse with $a = 6$ and $b = 3$. The steps are as follows :

1. input the ellipse equations:
$$L_1: \frac{x^2}{6^2} + \frac{y^2}{3^3} = 1 \text{ and } L_2: \frac{x^2}{3^2} + \frac{y^2}{6^3} = 1;$$
2. place point A on L_1 and point B on L_2;
3. compute distance: OA=(Distance(O, A), OB = Distance(O, B);
4. calculate the scaling factor k:
$\mathrm{k} = (\mathrm{Distance}(\mathrm{O}, \mathrm{A}) + \mathrm{Distance}(\mathrm{O}, \mathrm{B}))/(2 * \mathrm{Distance}(\mathrm{O}, \mathrm{A});$
5. determine the coordinates of point C: $\mathrm{C} = \mathrm{O} + \mathrm{k} * (\mathrm{A} - \mathrm{O});$
6. enable the trace for point C to visualize its path;
7. move point B along the ellipse trajectory.

By following these steps, the resulting illustration corresponds to Figure 3. The interactive GeoGebra file used to create this figure can be accessed at the following link: https://www.geogebra.org/calculator/mhucd84t. You can click point B and drag it around the ellipse; as you do so, point C will trace out a curve that represents the mean distance function.

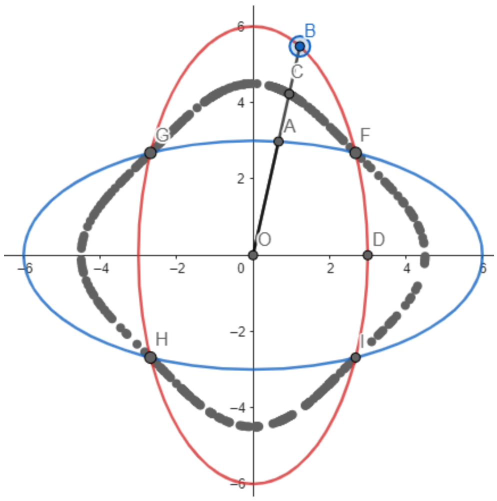


Figure 3. Visualizing the curve $\mathbf{g(t)}$ as a closed curve. The curve $\mathbf{g(t)}$ closely resembles a superellipse. When rotated by $\boldsymbol{\pi/4}$ relative to O, its shape nearly matches the sketch proposed by Pfiefer in Figure 1, except for a scaling factor of $\mathbf{1/2}$.

2.1.5 Parametric Equations with Variable Radius

Theorem 2.3. The parametric equations for a closed curve with a variable radius $g(t)$ are given by:

$$x' = ka\cos(t), \quad y' = kb\sin(t),$$

where $k = (p+q)/2q$ with

$$p = \sqrt{a^2\cos^2(t) + b^2\sin^2(t)} \text{ and } q = \sqrt{a^2\sin^2(t) + b^2\cos^2(t)}.$$

Proof. We seek to derive the parametric equations for the curve $g(t)$, as illustrated in Figure 3. Suppose there are two ellipses centered at the origin $O(0,0)$. Let point A lie on the first ellipse, with coordinates $A(a\cos(t)\,, b\sin(t))$. The distance from the origin to this point is:

$$p = OA = \sqrt{a^2\cos^2(t) + b^2\sin^2(t)}.$$

Let point B lie on the second ellipse, with coordinates$B(b\cos(t)\,, a\sin(t))$. Its distance from the origin is

$$q = OB = \sqrt{a^2\sin^2(t) + b^2\cos^2(t)}.$$

Now, consider a point $C(x', y')$ that lies on the straight-line connecting A and B. such that it also lies along the direction of OA. The distance from the origin to C is defined as the average of the distances OA and OB, namely

$$OC = g(t) = \frac{1}{2}(OA + OB) = \frac{1}{2}(p+q).$$

Since C lies on the ray from O through A, its coordinates can be written as:

$$C = O + k(A - O) = k \cdot A = (ka\cos(t), kb\sin(t))$$

where

$$k = \frac{OC}{OA} = \frac{p+q}{2p}$$

to simplify the resulting parametric form of point C, leading to

$$x' = ka\cos(t), \quad y' = kb.$$

Therefore, we obtain the parametric equations for the closed curve with variable radius $g(t)$, as stated in the theorem. ∎

2.1.6 A New Lower Bound for the Ellipse Perimeter

Theorem 2.4. The perimeter of an ellipse satisfies the following lower bound:

$$\pi\left(\frac{a+b}{2}+\sqrt{\frac{a^2+b^2}{2}}\right)\leq L(a,b)$$

where $a \geq b \geq 0$.

Geometric Analysis

One of the most important aspects of bounding the perimeter of an ellipse is determining a strong lower bound. A key challenge is determining whether a lower bound sharper than $(a+b)/2$ can be found. Referring to Figure 2, the approach involves drawing a line from the lower bound point $(a+b)/2$ to the upper bound point $\sqrt{(a^2+b^2)/2}$. Then, a vertical line is drawn perpendicular to the x-axis through this boundary point. As a result, a trapezoidal region is formed, bounded by the perpendicular line, the x-axis, the y-axis, and the line connecting the two boundary points.

The area of this trapezoid is:

$$L_A=\frac{1}{8}\pi\left(\frac{a+b}{2}+\sqrt{\frac{a^2+b^2}{2}}\right).$$

This area provides an approximate estimate for the integral:

$$\int_0^{\frac{\pi}{4}} f(t)\ dt.$$

If the trapezoidal rule with $n \geq 2$ subintervals is used, a larger lower bound cannot be obtained. Thus, we assume:

$$L_A=\frac{1}{8}\pi\left(\frac{a+b}{2}+\sqrt{\frac{a^2+b^2}{2}}\right)\approx\int_0^{\frac{\pi}{4}} f(t)\ dt.$$

This approach requires further investigation because for low eccentricity, the function $f(t)$ exhibits varying concavity, causing the area above and below the trapezoid to be unequal. However, for high eccentricity $e \to 1$, the function $f(t)$ becomes entirely convex. This suggests that:

$$L_A \leq \int_0^{\frac{\pi}{4}} f(t)\ dt.$$

Geometrically, this indicates that L_A represents the largest possible lower bound.

The equation of the line connecting the extremal points is

$$y = \frac{y_2 - y_1}{x_2} x + y_1$$

where $y_1 = (a+b)/2$, $y_2 = \sqrt{(a^2+b^2)/2}$, $x_2 = \pi/4$, and $x = t$. Thus,

$$y = \frac{4}{\pi}\left(\sqrt{\frac{a^2+b^2}{2}} - \frac{a+b}{2}\right)t + \frac{a+b}{2}.$$

Finding the intersection of y and $f(t)$ algebraically is difficult, but graphical analysis shows an intersection for $e < 1$. Figure 4 illustrates how this intersection shifts as the eccentricity decreases from 1 to 0. It is observed that for $e < 1$, the area below the line (which is computed) is smaller than the area above the line (which is not accounted for). As e decreases, this difference diminishes.

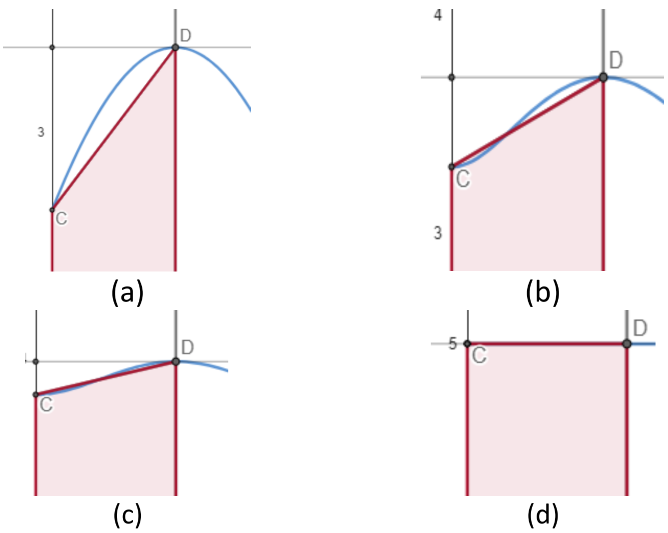


Figure 4. The figure illustrates the approximation of $\mathbf{L(a,b)/8}$ using the trapezoidal method. The line CD connects the lower and upper bound points for various eccentricities. For $\mathbf{e < 1}$, the area under this line underestimates the actual area under the curve, but as $\mathbf{e}$ decreases, the discrepancy diminishes, indicating improved accuracy of the approximation. Figure (a) shows the case for $\mathbf{e = 1}$; (b) for $\mathbf{e < 1}$; (c) for $\mathbf{e \ll 1}$; and (d) for $\mathbf{e = 0}$, where the approximation becomes exact.

Based on this geometric analysis, we conclude that there exists another tight lower bound

$$\pi\left(\frac{a+b}{2}+\sqrt{\frac{a^2+b^2}{2}}\right)\leq L(a,b).$$

We can prove Theorem 2.4 using reduction to a single variable and derivative analysis. Additionally, Barnard (2001) proved the following inequality:

$$2\pi\left(\frac{a^{\frac{3}{2}}+b^{\frac{3}{2}}}{2}\right)^{\frac{2}{3}}< L(a,b)$$

(Barnard et al., 2001). Now, we need to prove that:

$$\pi\left(\frac{a+b}{2}+\sqrt{\frac{a^2+b^2}{2}}\right)\leq 2\pi\left(\frac{a^{\frac{3}{2}}+b^{\frac{3}{2}}}{2}\right)^{\frac{2}{3}}< L(a,b.)$$

Since this is a homogeneous inequality (both sides have order 1), we normalize by setting $b=1$ and $a=t\geq 1$). Substituting, we obtain:

$$\left(\frac{t+1}{2}+\sqrt{\frac{t^2+1}{2}}\right)\leq 2\left(\frac{t^{3/2}+1}{2}\right)^{2/3}.$$

We can simplify the inequality above into the form

$$t+1+\sqrt{2(t^2+1)}\leq 2^{\frac{4}{3}}\left(t^{3/2}+1\right)^{2/3}.$$

Define the function:

$$g(t)=2^{\frac{4}{3}}\left(t^{3/2}+1\right)^{2/3}-\left(t+1+\sqrt{2(t^2+1)}\right).$$

We aim to show that $g(t)\geq 0$, for all $t\geq 1$. The proof proceeds as follows:

Continuity of $g(t)$ on $[1,\infty)$

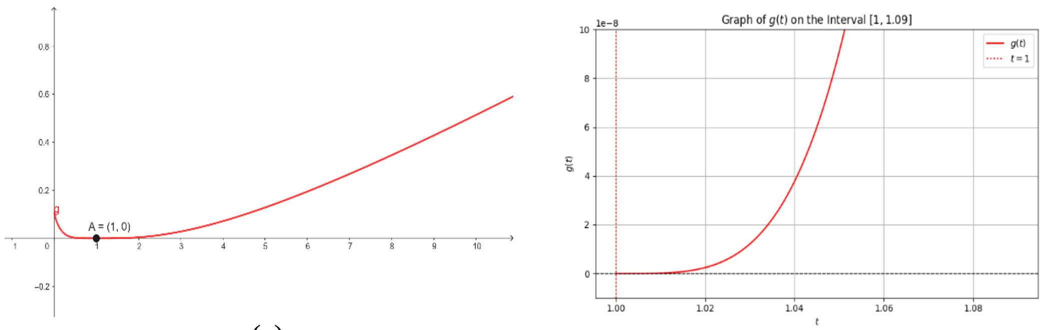
The function

$$g(t)=2^{\frac{4}{3}}\left(t^{3/2}+1\right)^{2/3}-\left(t+1+\sqrt{2(t^2+1)}\right)$$

is defined on $[0,\infty)$. In this analysis, we focus on the interval $[1,\infty)$. Since $g(t)$ is composed of polynomials, square roots, and rational exponents, all of which are continuous on $[1,\infty)$, it follows that $g(t)$ is also continuous on this interval.

Monotonicity of $g(t)$ on $[1, \infty)$

As shown in Figure 5(a), the graph of $g(t)$ intersects the t-axis exactly once at the point $A(1,0)$, confirming that $g(1) = 0$. The graph also clearly illustrates that $g(t)$ is increasing on the interval $[1, \infty)$ and remains nonnegative throughout this domain. Moreover, $\lim_{t \to 1^+} g(t) = 0^+$ and $\lim_{t \to +\infty} g(t) = +\infty$ indicate that $g(t)$ approaches zero from above near $t = 1$, and increases without bound as t becomes larger. To emphasize this behavior, Figure 5(b) shows a zoomed-in version of the graph over the interval $[1, 1.09]$, clearly showing that $g(t)$ is indeed increasing in that region.



(a) (b)

Figure 5. (a) The function $\mathbf{g(t)}$ is decreasing on $[\mathbf{0}, \mathbf{1})$ and increasing on $(\mathbf{1}, \infty)$, with $\mathbf{g(1) = 0}$ (generated using GeoGebra), and (b) A zoomed-in plot of $\mathbf{g(t)}$ on the interval $[\mathbf{1}, \mathbf{1.09}]$, showing the increasing nature of the function (generated using Python).

Since $g(t)$ is continuous on the interval $[1, \infty)$, $g(1) = 0$, and $g(t)$ is increasing on $[1, \infty)$, it follows that $g(t) \geq 0$ for all $t \geq 1$.

2.2 Discussion

2.2.1 Bounds of Ellipse Perimeter

After establishing the upper and lower bounds for the perimeter of an ellipse, the next step is to determine the greatest lower bound. Finding this bound is not trivial, as it requires rigorous proof. As is known, applying the trapezoidal rule with $n = 1$, yields a lower bound greater than $\pi(a + b)$, namely

$$L(a, b) = \frac{1}{2}\pi\left((a + b) + \sqrt{\frac{a^2 + b^2}{2}}\right).$$

as shown in Figure 4(a). However, verifying this bound requires a rigorous justification for all values of eccentricity. Visually, as shown in Figure 4, it appears possible to establish a lower bound that is greater than $\pi(a+b)$. In Figure 4, a trapezoid can be constructed with parallel sides passing through the extreme points of the function $f(t)$, where its area closely approximates the area under the curve $f(t)$.

This trapezoidal area could serve as a candidate for a new upper bound on the integral $L(a,b)$. Based on Figure 4, we can construct a trapezoid whose pair of parallel sides pass through the extreme points of $f(t)$ over the interval $[0,\pi/4]$. By adjusting its dimensions, we can ensure that its area remains close to $1/8$ $(L(a,b)$ but slightly above it. Observing the region bounded by $f(t)$ and the line $y=\sqrt{(a^2+b^2)/2}$ we see that it provides enough area to form the desired trapezoid. The area of this trapezoid lies between $\pi/8(\sqrt{(a^2+b^2)/2})$ and $1/8$ $(L(a,b)$.

One approach that requires further verification is replacing the ellipse with a circle whose perimeter is approximately the same. This circle has an average radius derived from a and b (Lehmer, 1950; Rohman, 2022), namely

$$r(a,b)=\left(\frac{1}{2}\left(a^{\lambda}b^{\mu}+a^{\mu}b^{\lambda}\right)\right)^{q}\quad (q^{-1}=\lambda+\mu). \tag{5}$$

For example, for $\lambda=2$ d and $\mu=0$ then $q^{-1}=\lambda+\mu=2$ or $q=1/2$, then substitute into equation (5) we get

$$r(a,b)=\left(\frac{1}{2}(a^2+b^2)\right)^{1/2}.$$

Furthermore, the perimeter of the ellipse can be approximated as

$$L(a,b)=R=2\pi r(a,b)=\sqrt{2}\pi(a^2+b^2)^{1/2}$$

as shown in the table 1.

Table 1 presents several well-known approximations for the perimeter of the ellipse; each derived from a specific form of the function $r(a,b)$.

Table 1. Approximation $\boldsymbol{L(a,b)=2\pi r(a,b)}$

Approximation: $\boldsymbol{L(a,b)=2\pi r(a,b)}$	$\boldsymbol{(\lambda,\mu)}$
$R=\sqrt{2}\pi(a^2+b^2)^{1/2}$	(2,0)
$M=\pi(a+b)$	(1,0)
$G=2\pi(ab)^{1/2}$	(1,1)
$H_1=4\pi(ab)/(a+b)=G^2/M$	(-1,0)
$H_2=\sqrt{8}\pi(ab)(a^2+b^2)^{-1/2}=G^2/R$	(-2,0)

Linear combinations, such as $(M+R)/2$ or $(3M-G)/2$, have been explored and discussed in Theorem 2.4.

Since it has been proven that $M\le L(a,b)\le R$, other approximations deviate significantly and are not suitable as bounds for $L(a,b)$. Thus, an appropriate choice for $r(a,b)$ should satisfy

$$R\le 2\pi r(a,b)\le M. \tag{6}$$

This inequality holds if we choose $1<\lambda<2$ and $\mu=0$, or use linear combinations such as $(M+R)/2$. For example, setting $2\pi r(a,b)$ for $\lambda=\frac{3}{2},\mu=0,q=2/3$ leads to Muir's approximation. While $(M+R)/2$ is correlated with Theorem 2.4.

One well-known result providing a lower bound for $L(a,b)$ is the Muir approximation (1883):

$$L(a,b)>2\pi\left(\frac{a^{3/2}+b^{3/2}}{2}\right)^{2/3}.$$

This approximation was later reinforced by Vuorinen (1997) as a lower bound and formally proven by Barnard (2000) using a generalization of the hypergeometric function (Barnard et al., 2001; M. Wang et al., 2012).

Using the inequality $\pi(a+b)\le L(a,b)$ and the Power Mean Inequality (PMI), we know that for $r=3/2>\ s=1$, the following holds

$$\left(\frac{a^r+b^r}{2}\right)^{\frac{1}{r}}\ge\left(\frac{a^s+b^s}{2}\right)^{\frac{1}{s}}$$

(Alsina & Nelsen, 2009; Barbeau, 2016). Applying PMI, we obtain:

$$\pi(a+b) \leq \pi\left(\frac{a+b}{2} + \sqrt{\frac{a^2+b^2}{2}}\right) \leq 2\pi\left(\frac{a^{\frac{3}{2}}+b^{\frac{3}{2}}}{2}\right)^{\frac{2}{3}} \leq L(a,b)$$

which aligns with the illustration in Figure 4, where the lower bound of $L(a,b)$ is depicted.

A second approach, developed by Pain (2022), applies an integral mean-based method similar to the work of Gusić and Jameson to reinforce the inequality $\pi(a+b) \leq L(a,b)$. In his paper, Pain defines

$$L = L_1 = L_2 = \frac{1}{2}(L_1 + L_2) \geq (a+b)\frac{\pi}{4}$$

where $L_1 \geq \int_0^{\pi/2} \left(a\cos^2(\phi) + b\sin^2(\phi)\right)d\phi$ and $L_2 \geq \int_0^{\pi/2} \left(b\cos^2(\phi) + a\sin^2(\phi)\right)d\phi$.

By setting $ab = R^2$, the minimum of $a+b$ given by $f(a) = a + R^2/a$ is achieved for $R = \sqrt{ab}$, yielding the well-known result:

$$\frac{\pi}{2}\sqrt{ab} \leq L$$

For the total perimeter of the ellipse, this leads to $\pi\sqrt{ab} \leq L_e$

Moreover, observing that $L = L_1 = L_2 = \frac{1}{2}(L_1 + L_2) \geq \pi(a+b)/4$, where L_1 and L_2 are integrals over the interval $[0, \pi/4]$, we obtain

$$4\frac{(a+b)\pi}{4} \leq 4L = L_e.$$

Simplifying, we get $(a+b)\pi \leq 4L = L_e$, which is consistent with Theorem 2.2 and the results of Jameson, Gusić, and Pfiefer.

One of the most well-known and accurate formulas for the ellipse perimeter is Ramanujan's second approximation:

$$P \approx \pi(a+b)\left(1 + \frac{3h}{10 + \sqrt{4-3h}}\right), \text{where } h = \frac{(a-b)^2}{(a+b)^2}.$$

Numerical calculations suggest that P is very close to $L(a,b)$ and satisfies $P \leq L(a,b)$ (Almkvist & Berndt, 1988; B. Villarino, 2008). However, a rigorous proof establishing P as the greatest lower bound remains an open problem.

1.2.2 The Greatest Lower Bound of the Ellipse Perimeter

We propose a conjecture that the greatest lower bound of the perimeter of an ellipse is given by

$$L^* = \pi\left(\frac{a+b}{2} + \sqrt{\frac{a^2+b^2}{2}}\right).$$

This conjecture is based on several key observations. The lower bound was derived using the trapezoidal rule with a single subinterval (i.e., $n = 1$) applied over the interval $[0, \pi/4\,]$.

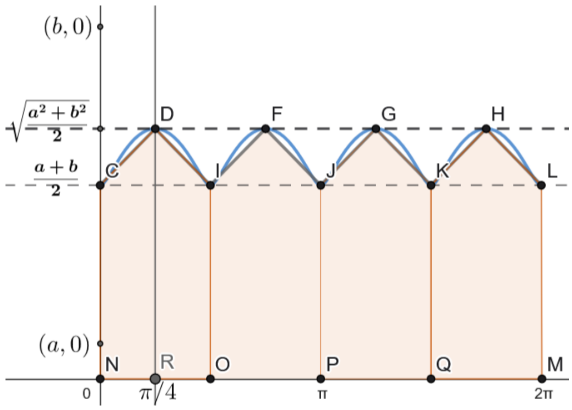


Figure 6. Trapezoidal approximation for computing $\mathbf{L(a,b) = 8\int_0^{\pi/4} f(t)\,dt}$. Observe that when $\mathbf{L(a,b)/8 = L^*/8}$ using the trapezoidal rule with $\mathbf{n = 1}$ over the interval $[\mathbf{0}, \boldsymbol{\pi}/\mathbf{4}]$, we find that $\frac{\mathbf{L^*}}{\mathbf{8}} > \frac{\boldsymbol{\pi}}{\mathbf{4}}\left(\frac{\mathbf{a+b}}{\mathbf{2}}\right)$ which implies that $\mathbf{L^*} > \mathbf{2}\boldsymbol{\pi}\left(\frac{\mathbf{a+b}}{\mathbf{2}}\right)$ over the interval $[\mathbf{0}, \mathbf{2}\boldsymbol{\pi}]$.

To analyze whether L^* is indeed the greatest lower bound, we must test it across all values of eccentricity. Geometrically, as shown in Figure 4(a), when the eccentricity is $e = 1$ the integrand is convex over the interval $[0, \pi/4]$ This allows us to ensure that

$$\frac{L(a,b)}{8} = \frac{L^*}{8} > \frac{\pi}{4}\left(\frac{a+b}{2}\right)$$

which implies that

$$L^* > 2\pi\left(\frac{a+b}{2}\right)$$

over the interval $[0, 2\pi]$. However, in Figures 4(b) and 4(c), the function is no longer convex, as the line CD intersects the curve $f(t)$ for $0 < e < 1$. Figure 5 further confirms this behavior by showing that the line CD intersects $f(t)$ whenever $0 < e < 1$.

Note that $L^*/8$ serves as a lower bound for the ellipse perimeter over the interval $[0, \pi/4]$, Hence, the perimeter can be expressed as

$$\frac{L(a,b)}{8} = \frac{L^*}{8} + |E_1|$$

where

$$|E_1| = \frac{\pi}{4^3.12} f''(\xi)$$

for some ξ in the interval of integration.

While deriving the second derivative $f''(t)$ is analytically complex, numerical computations for $a = 3$ and $b = 2$ ((i.e. $0 < e < 1$) show that $f''(t)$ takes positive values at several points within the interval $[0, \pi/4]$, as illustrated in Figure 7. This allows us to establish a positive value for $|E_1|$, which in turn confirms that

$$\frac{L^*}{8} > \frac{\pi}{4}\left(\frac{a+b}{2}\right)$$

or equivalently,

$$L^* > 2\pi\left(\frac{a+b}{2}\right)$$

over the interval $[0, 2\pi]$.

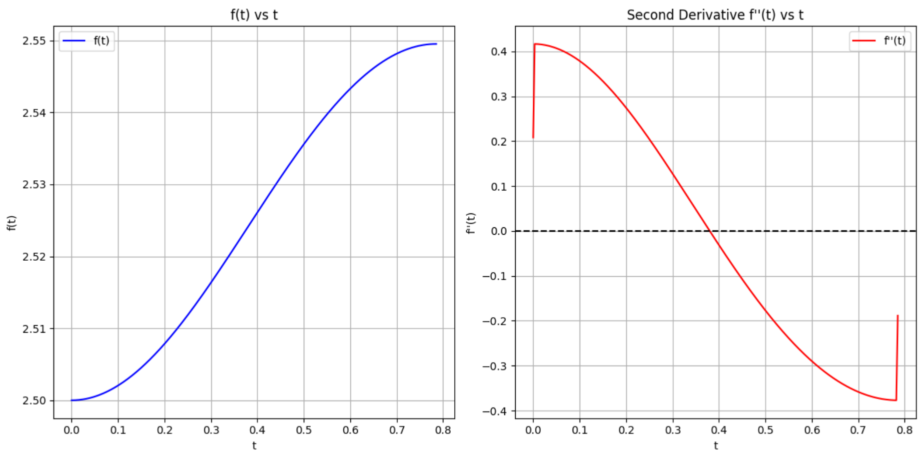


Figure 7. The functions $\mathbf{f(t)}$ and $\mathbf{f''(t)}$ for $\mathbf{a = 3}$ and $\mathbf{b = 2}$. Although $\mathbf{f(t)}$ is not convex over the interval $[\mathbf{0}, \boldsymbol{\pi}/\mathbf{4}]$, there exist positive values of $\mathbf{f''(t)}$ within this interval. These positive values play a crucial role in estimating the error term $|\mathbf{E_1}|$

1.2.3 Validation of the Curve for the Mean Integral Function

The graph of the mean integral function, as shown in Figure 3, is proposed to represent the ellipse perimeter $L(a,b)$. To validate this, we estimate the curve length using geometric approximations. The curve's length can be approximated geometrically using shapes such as rectangles or rhombuses, as illustrated in Figure 8.

By drawing four tangents at the intersection points between the mean curve and the ellipse, we form an outer rhombus that provides an upper bound:

$$4F_2F_7 = 4\sqrt{\frac{a^2+b^2}{2}}. \tag{7}$$

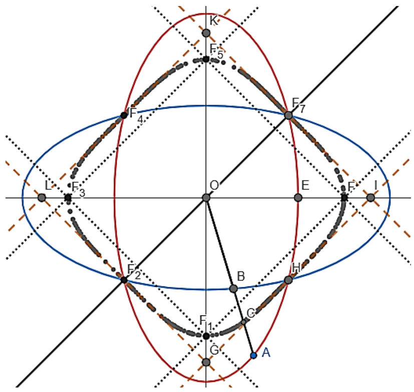


Figure 8. Geometric approximation of the ellipse perimeter using the curve of the mean integral function

A smaller rhombus formed through intersections with the x- and y-axes gives a lower bound:

$$4FF_5 = 4\sqrt{\left(\frac{a+b}{2}\right)^2+\left(\frac{a+b}{2}\right)^2} = 2\sqrt{2}(a+b). \tag{8}$$

Combining these with Theorem 2.2 gives that although both $\pi(a+b)$ and $2\sqrt{2}(a+b)$ serve as lower approximations of $L(a,b)$, we must be cautious with inequality chains. Since $\pi > 2\sqrt{2}$, it follows that

$$2\sqrt{2}(a+b) \le L(a,b) \le 4\sqrt{\frac{a^2+b^2}{2}} \le 2\pi\sqrt{\frac{a^2+b^2}{2}}.$$

This emphasizes that $\pi(a+b)$ is a tighter lower bound compared to $2\sqrt{2}(a+b)$ despite both being linear in form.

3. CONCLUSION AND SUGGESTIONS

This study refines the integral representation of the ellipse perimeter by utilizing the extreme points of the function as integral limits:

$$L(a,b) = 8\int_0^{\frac{\pi}{4}} f(t)\ dt$$

where

$$f(t) = \frac{1}{2}\left(\sqrt{a^2\cos^2(t) + b^2\sin^2(t)} + \sqrt{a^2\sin^2(t) + b^2\cos^2(t)}\right).$$

The function $f(t)$ exhibits eight-fold symmetry over the interval $[0, 2\pi]$. Its graph reveals clear extreme points, which reflect the behavior of the integrand associated with the ellipse perimeter. These graphical features can be used to estimate potential lower and upper bounds of $L(a,b)$ with greater precision.

In numerical computations, this reformulated integral tends to produce smaller errors compared to the standard elliptic integral of the second kind. This improvement is due to the shorter integration interval $[0, \pi/4]$, which reduces the accumulation of numerical errors typically encountered in more extended or complex intervals. For instance, when applying the trapezoidal rule, the approximation error E is given by

$$E = -\frac{(\pi/4)^3}{12N^2} f''(\xi)$$

which is one-eighth of the error obtained when using the full interval $[0, \pi/2]$.

To further enhance our understanding of the ellipse perimeter, future research should focus on investigating the tightest possible lower and upper bounds based on this integral. Comparative studies with classical elliptic integrals and established formulas, such as Ramanujan's second approximation, will be essential to evaluate its effectiveness and accuracy.

Future research could also explore the curve of the mean integral function to determine new bounds for the ellipse perimeter. For instance, I identified a potential new upper bound—equation (7)—which requires further analytical proof to confirm its validity. Meanwhile, the lower bound in equation (8) is analytically proven to be smaller than the bound established by Theorem 2.3.

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