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IDENTITY GRAPH OF GENERALIZED QUATERNION GROUP

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ABSTRACT. *Identity graphs are graphs defined based on the relationship of group elements to identity elements. This study examines the relationship between the algebraic structure of the generalized quaternion group. Through a review of literature related to graph theory and group theory, this study identifies patterns in the identity graph of the generalized quaternion group. The results show that the identity graph formed has distinctive characteristics, such as a specific number of edges and cycles, as well as varying degrees of each vertex. Furthermore, this study proves that the identity graph of the generalized quaternion group is neither an Euler graph nor a Hamiltonian graph. Moreover, based on the pattern formed, the identity graph on the group always forms a planar graph. The discussion is closed with the identification of its girth and diameter.*

Keywords*: Identity Graph, Generalized Quarternion Group, Euler, Hamilton, Planar*

ABSTRAK. *Graf identitas adalah graf yang didefinisikan berdasarkan hubungan elemen grup dengan elemen identitas. Penelitian ini mengkaji hubungan struktur aljabar dari grup quaternion tergeneralisasi. Melalui tinjauan literatur yang berkaitan dengan teori graf dan teori grup, dapat diidentifikasi pola-pola pada graf identitas dari grup quaternion tergeneralisasi. Hasil penelitian menunjukkan bahwa graf identitas yang terbentuk memiliki karakteristik yang khas, seperti jumlah edge dan cycle tertentu dan derajat yang unik pada setiap vertexnya. Lebih lanjut, penelitian ini membuktikan bahwa graf identitas dari grup quaternion tergeneralisasi bukan merupakan graf Euler maupun graf Hamiltonian. Selain itu, berdasarkan pola yang terbentuk, graf identitas pada grup tersebut selalu membentuk graf planar. Pembahasan ditutup dengan identifikasi eksentrisitas dan diameternya.*

Kata kunci: Graf Identitas, Grup Quarnion Tergeneralisasi, Euler, Hamilton, Planar

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1. INTRODUCTION

Graph theory and group theory are two branches of mathematics that have developed significantly and have wide applications in various fields, such as computer science, physics, chemistry, and network theory. One of the interesting connections between these two branches is the use of groups in constructing and analyzing graphs, where group structures can provide deep insights into the properties of graphs. Representation of groups on graphs is a topic that is currently being widely researched. It connects two fields of Mathematics, namely graph theory and group theory. Graphs are used to represent discrete objects and the relationships between these objects allowing discrete objects and their relationships to be visualized. The relationship between group theory and graphs was first introduced by Artur Cayley in 1878 (Yamasaki, 2018). Other researches that discuss Cayley graph include (Tamizh Chelvam et al., 2020) and (Huang et al., 2022). Group representations on other types of graphs can be found in research (Munandar, 2023) and (Shofiyulloh et al., 2024).

There are several types of groups in group theory in mathematics. Some examples of groups include the dihedral group and the quarternion group (Bariyah, 2020). The quaternion group, which is one of the non-Abelian groups with eight elements, has an important role in group theory and its applications (Gazir S & Wardhana, 2021). This group has unique properties, such as non-commutative elements, which allow the formation of graphs with special structures and characteristics. In this context, identity graphs, Euler graphs, and Hamiltonian graphs are three relevant graph types worth further exploration.

The identity graph is defined based on the relationship between a group element and the identity element. The representation of groups on graphs in the form of identity graphs was first introduced in (Kandasamy & Smarandache, 2009). Some subsequent studies that developed it can be found in (Herawati, 2022) and (Vianney et al., 2021). These graphs are often used to understand the symmetries and fundamental relationships between group elements. On the other hand, Euler graphs and Hamiltonian graphs focus on certain paths and cycles in the graph. Euler graphs refer to graphs that have paths or cycles that pass through each edge exactly once, while Hamiltonian graphs are concerned with paths or cycles that pass through each vertex exactly once

Research (Muftirridha & Hidayat, 2016) discusses the identity graph of the dihedral group. This research will look at the representation of the identity graph on the generalized quaternion group. By identifying the patterns of the graph formed, it can be found that some results bind to the identity graph of the generalized quaternion group. Through the patterns that appear, the Hamiltonian, Eulerian and Planarity can be identified. Then there are also results related to the diameter and girth of the graph formed. To the best of the author's knowledge, no previous research has investigated this matter.

2. METHODS AND THEORETICAL FOUNDATION

This research is conducted by reviewing various relevant references to support the discussion of identity graphs over generalized quaternion groups Q_{4n}. The research process includes reading, understanding, and analyzing the characteristics of the group Q_{4n}, including the identity graph, eulerian graph, Hamiltonian graph, and other related properties. By paying attention to the patterns of the identity graph over the generalized quaternion group Q_{4n}, the properties of the identity graph can be found.

Terminology regarding graphs refers to the source (Augustin & Welyanti, 2020) while terminology related to generalized quaternion groups refers to the source (Munandar, 2023)

Definition 2.1. (Augustin & Welyanti, 2020) Given G is a group with identity element e and $x, y \epsilon G$. The identity graph of G denoted Γ_G is a graph whose vertices are elements in G and two vertices $x, y \epsilon G$ with $x \neq y$ are connected if $xy = e$.

Definition 2.2. (Augustin & Welyanti, 2020) A connected multigraph G is called an Euler multigraph if it contains a circuit that contains all edges in the graph G. The circuit that causes a multigraph to be Euleran is called an Euler circuit.

Theorem 2.3. (Augustin & Welyanti, 2020) A connected multigraph G is Eulerian if every vertex in G has even degree.

Definition 2.4. (Augustin & Welyanti, 2020) A connected graph$G$ is called a Hamiltonian graph if it contains a cycle (closed walk) that consist of all vertices in the graph G with no repeated vertices. The cycle that causes the graph to be Hamiltonian is called a Hamiltonian cycle.

Theorem 2.5. (Munandar, 2022) Given a graph G. If G is a Hamiltonian graph, then for any nonempty set $S \subseteq V(G)$ the component of $G - S$ are less or equals than $|S|$.

Definition 2.6. (Munandar, 2023) The generalized quaternion group (Q_{4n}) is an extension group of the quaternion group structure with the number of elements depending on $n \geq 2$, defined as follows:

$$Q_{4n} = \langle a, b | a^{2n} = e, b^2 = a^n, b^{-1}ab = a^{-1} \rangle.$$

Definition 2.7. (Munandar, 2023) Suppose that Q_{4n} is a generalized quaternion group. The order of each element in Q_{4n}

$$o(a^i b^j) = \begin{cases} \dfrac{2n}{gcd(i, 2n)}, & j = 0 \\ 4 \quad , & j = 1 \end{cases}$$

Definition 2.7. (Munandar, 2022) Given a multigraph G with $V(G) = \{v_i | i = 1,2, \dots n\}$ is the set of vertices in G. The degree of each vertex $v_i \in V(G)$ denoted $der(v_i)$ is the number of edges incident to that vertex.

Definition 2.8. (Augustin & Welyanti, 2020) Given a graph G with $|V(G)| \geq 3$. The graph G is called a cycle graph if$V(G) = \{v_1, v_2, \dots, v_n\}$ and $E(G) = \{v_1v_2, v_2v_3, v_3v_4, \dots, v_nv_1\}$. The cycle graph with order n is denoted by C_n.

Theorem 2.9. (Augustin & Welyanti, 2020) A graph is a planar graph if and only if it contains no subgraphs of complete bipartite $K_{3,3}$ or and complete graph K_5.

Definition 2.10. (Abdussakir, 2017) The girth of a G graph is the minimum length of a vertex contained in the G and denoted by (G). If there is no cycle formed in the graph G then $(G) = \infty$.

Definition 2.11. (Munandar, 2022) Given x and y distinct vertices in the graph . The distance from x to y is the length of the shortest path in the graph G from vertex x to vertex y and denoted by $d(x, y)$. If there is no path between vertex x to y then $d(x, y) = \infty$.

Definition 2.12. (Abdussakir, 2017) Given a connected multigraph G with $v \in (G)$. The eccentricity of G is the distance from v to the farthest vertex in G and denoted by (v).

Definition 2.13. (Ma, 2016) The diameter of a graph G is the maximum eccentricity at each vertex in G denoted by $di(G)$.

3. RESULTS AND DISCUSSION

This section presents an example of the identity graph of the generalized quaternion group Q_{12}. This example is chosen as it effectively illustrates the structure and characteristics of identity graphs relevant to the discussion. Through observation of the identity graph of Q_{12}, certain patterns can be identified and used as a basis to formulate theorems that explain the properties of identity graphs over generalized quaternion groups Q_{4n} in general.

Example 3.1. Given a quaternion group Q_{12}. With

$$Q_{12} = \{a, a^2, a^3, a^4, a^5, e, b, ab, a^2b, a^3b, a^4b, a^5b\}.$$

Based on the definition of the identity graph, the identity graph of Q_{12} is

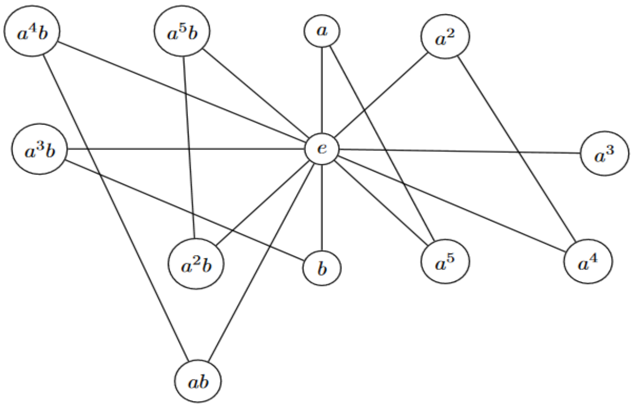

Figure 1. Identity Graph of Q_{12}

Based on Example 3.1 above, it can be observed that the identity graph for Q_{12} is five cycle graph C_3 and an edge. In general, this result can occur for any quaternion group as written in the following theorem

Theorem 3.2. Given$\Gamma_{Q_{4n}}$ is the identity graph over the generalized quaternion groupQ_{4n}, then $\Gamma_{Q_{4n}}$ has $(2n-1)C_3$ and an edge.

Proof. Based on the patterns observed in Example 3.1, it can be found that for any natural numbers n, the graph formed will form the cycle pattern C_3 as follows:

$$e - a^i b - a^{i+n} b - e, \text{ with } i = 0, 1, 2, \dots, n-1$$

$$e - a^j - a^{2n-j} - e, \text{ with } j = 1, 2, \dots, n-1.$$

Thus, the number of cycles C_3 formed from the identity graph over the generalized quaternion group Q_{4n} is $2n-1$. Furthermore, for every vertex a^n, since

$$a^n \cdot a^n = a^{2n} = e,$$

then the vertex a^n is a *self-inverse* element, so based on the definition of identity graph, the element a^n is only connected to the identity element e. In other words, the formed graph also has an edge connecting e and a^n. So, the identity graph $\Gamma_{Q_{4n}}$ has $(2n-1)$ *cycle* C_3 and an edge. ■

Theorem 3.3. If $\Gamma_{Q_{4n}}$ is the identity graph over the generalized quaternion group Q_{4n}, then the degree of each element of the group is

$$deg(a^i b) = deg(a^j) = 2, deg(a^n) = 1 , deg(e) = 4n - 1,$$

with, $i = 0, 1, 2, \dots, 2n - 1$ and $j = 1, 2, \dots, 2n - 1, j \neq n$.

Proof. It is clear that the vertex$e$ is connected to all elements of the group, so $\text{der}(e) = 4n - 1$. Then for any natural number n, there is a subgroup constructed by the elements of a , viz:

$$G_1 = \langle a \rangle = \{a, a^2, a^3, \dots, e\}$$

Every element of G_1 will be connected to identity element e. In addition, since $a^j \cdot a^{2n-j} = e$, with $= 1, 2, \dots, n - 1$, then every vertex a^j is also connected with vertex a^{2n-j}. So, for every vertex a^j the degree is 2, except for vertex a^n. Because

$$a^n \cdot a^n = a^{2n} = e,$$

then the vertex a^n is a *self-inverse* element. Based on the restriction in the definition of identity graph, the element a^n is not connected to itself. Consequently, for each vertex a^n is only connected to the identity e, It is mean this vertex has degree 1.

Furthermore, based on the above information, each vertex $a^i b$ is connected to the identity element e. In addition, since $a^i b \cdot a^{i+n} b = e$, with $i = 0, 1, 2, \dots, n - 1$, then each vertex $a^i b$ is also connected to with vertex $a^{i+n} b$. Thus, for every vertex $a^i b$ has degree two. ■

Theorem 3.4. For any natural number n, the identity graph over the generalized quaternion group $\Gamma_{Q_{4n}}$ is not an euler graph.

Proof. Since vertices a^n and e have degree 1 and $4n - 1$ respectively, then $\Gamma_{Q_{4n}}$ contains vertices with odd degree. Therefore, by Theorem 2.3 the identity graph $\Gamma_{Q_{4n}}$ is not an euler graph. ■

Theorem 3.5. For every natural number n, the identity graph over the generalized quaternion group $\Gamma_{Q_{4n}}$ is not a Hamiltonian graph

Proof. Note that vertex e is connected to all elements in identity graph $\Gamma_{Q_{4n}}$. By Theorem 3.2, vertexa^n has degree one. Choose$S = \{e\}$, then$\Gamma_{Q_{4n}} - S$ has$2n$

components. As a result, $c\left(\Gamma_{Q_{4n}} - S\right)$ is greater than $|S|$ for any natural number n. Thus, by Theorem 2.5 the graph $\Gamma_{Q_{4n}}$ is not a Hamiltonian graph. ■

Theorem 3.6. For any$n$ natural numbers, the identity graph of the generalized quaternion group $\Gamma_{Q_{4n}}$ is a planar graph.

Proof. Based on Theorem 3.2, the identity graph formed on the generalized quaternion group Q_{4n} is a cycle C_3 as much as $2n - 1$ with vertex center e, and an edge. Thus, the graph does not contain any subgraph $K_{3,3}$ or K_5. So, the identity graph over the group formed is a planar graph. ■

Since any vertex is connected with the identity element, the shortest path connecting any two vertices is two. This gives rise to the following theorem.

Theorem 3.7. For any n natural numbers, $diam\left(\Gamma_{Q_{4n}}\right) \leq 2$.

Proof. As a result of Theorem 3.2, the graph identity of Q_{4n} always connected to subgraph C_3. Because each subgraph always contains the identity vertex, then the diameter for every vertices that are not connected to each other is 2. ■

As a result of the fact that any vertex is always connected with the identity element, if two vertices x and y (no identity element in them) are connected to each other, they will form a cicle with the identity element. This is the give us the following theorem.

Theorem 3.8. Given a group Q_{4n}, then the girth of identity graph over Q_{4n} is 3 or ∞.

Proof. It is quite obvious with respect to Theorem 3.2. ■

4. CONCLUSIONS AND SUGGESTIONS

Based on the analysis of theoretical studies conducted by the author, it can be concluded that the identity graph on the generalized quaternion group has a unique structure. The patterns found in this study conclude that there is a connection between the algebraic properties of the generalized quaternion group and the topological properties of the identity graph. The identity graph of the generalized quaternion group Q_{4n} forms a certain pattern that is is neither an euler nor a hamilton graph. In addition, due to its unique shape, the graph formed is always a planar graph. The uniqueness also cause $diam(\Gamma_{Q_{4n}}) \leq 2$ and $g(\Gamma_{Q_{4n}})$ to be 3 or ∞.

Although this research successfully studied some characteristics of identity graphs, there are still many opportunities for further research. Future research can focus on generalizing the results of this study to other non-abelian groups, further analysis of the relationship between algebraic and topological properties of other theoretical characteristics.

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