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THE NEWTON-RAPHSON METHOD OF REAL-VALUED FUNCTIONS IN DISCRETE METRIC SPACE

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ABSTRACT. *This paper studies the Newton-Raphson method to approximate a root of a real-valued function in one-dimensional real discrete metric space. The method involves a derivative and is considered to be convergent very fast. However, the derivative is derived from the limit definition with respect to the Euclidean distance, different from that of the discrete metric space. This research investigates the Newton-Raphson method with respect to derivatives defined in discrete metric spaces by deriving the derivative first. The examined derivatives are absolute derivative and parameterised derivative on metric spaces. The results show that the constructed Newton-Raphson method can be an alternative root-finding method exemplified by some examples.*

Keywords*: Newton-Raphson method, discrete metric space, metric space derivative.*

ABSTRAK. Artikel ini membahas metode Newton-Raphson untuk mengaproksimasi akar dari fungsi bernilai riil pada ruang metrik diskrit riil berdimensi satu. Metode tersebut melibatkan turunan fungsi dan cenderung konvergen dengan cepat. Namun, turunan fungsi tersebut didapatkan dari definisi limit untuk jarak Euclid yang berbeda dengan jarak pada ruang metrik diskrit. Penelitian ini menyelidiki metode Newton-Raphson untuk turunan fungsi pada ruang metrik diskrit diawali dengan mengonstruksi turunan tersebut. Turunan yang digunakan adalah turunan mutlak dan turunan berparameter pada ruang metrik. Hasil penelitian menunjukkan bahwa metode Newton-Raphson yang dibentuk dapat menjadi metode alternatif untuk pencarian akar fungsi menurut beberapa contoh.

Kata Kunci: metode Newton-Raphson, ruang metrik diskrit, turunan ruang metric.

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1. INTRODUCTION

Numerical methods have been alternative approaches to solving mathematical problems that are hardly solvable by analytical methods. These problems often occur in real-world contexts, such as engineering and statistics (Chouhan & Gaur, 2022). One of the problems is root approximation for functions that are difficult to factorise. There are several numerical methods to approximate the roots of such functions, one of which is the Newton-Raphson method. The method involves derivatives to approximate a root of a function by iterations and was first used by Newton in 1669 (Epperson, 2013).

The Newton-Raphson is an open method meaning that it does not always converge. However, once convergent, it is very fast i.e. it converges to the root in fewer steps than slower root approximation methods (Epperson, 2013). Therefore, it can be a useful root approximation method applicable to solve such problems that are difficult to solve by analytical method.

The Newton-Raphson method involves a derivative resembling a tangent's gradient (Bartle & Sherbert, 2011) (the standard derivative). The tangent is used to approximate a root iteratively with an initial root approximation. The derivative is derived from the limit, whose definition involves Euclidean distance between two real numbers. However, there are many ways to study real numbers apart from viewing the numbers as a set with Euclidean distance. One of them is viewing such numbers as a metric space, a set together with a metric that generalises the concept of "distance" usually depicted by Euclidean distance. The new way to represent the distance can alter the limit, thus modifying the derivative. It then eventually disrupts the Newton-Raphson method, resulting in different iteration schemes.

This research investigates the Newton-Raphson method by modifying the derivative with respect to a different metric space that adjusts the standard derivative to a modified one. The metric space is a discrete metric space, a metric that results in either 0 or 1 that symbolises "same" and "different" respectively. Even though it rarely occurs in real-life problems (Kreyszig, 1978), it gives a unique perspective of distance, making it binary. Binary systems are apparent in computer or digital studies. Hence, studying this metric could be beneficial for development in such studies.

The derivatives of real-valued functions constructed in discrete metric space give potential Newton-Raphson iterations that can be alternative root-finding methods. The iterations may break some limitations in the standard Newton-Raphson methods and offer rapid convergence. Therefore, the methods could solve real-world problems that are presumably hard to calculate by the standard one.

2. DISCUSSION AND RESULTS

2.1 Preliminaries

We discover some definitions and theorems that help understand the results. The materials consist of the concepts of metric spaces, Banach spaces, limits and derivatives in the spaces, and the Newton-Raphson method.

Definition 2.1 (Metric Space). Let X be a set, and $d: X \times X \to \mathbb{R}$ be a function. The set X together with d, denoted by (X, d), is called a metric space if, for every $x, y, z \in X$, it satisfies the following conditions (Kreyszig, 1978):

i. Nonnegativity: $d(x, y) \geq 0$,
ii. $d(x, y) = 0$ if and only if $x = y$,
iii. Symmetry: $d(x, y) = d(y, x)$,
iv. Triangle inequality: $d(x, y) \leq d(x, z) + d(z, y)$.

The function d is called a metric.

Definition 2.2 (Normed Space). Let X be a vector space over real or complex numbers and $\|.\|: X \to \mathbb{R}$ be a function. The set X together with $\|.\|$, denoted by $(X, \|.\|)$, is called a normed space if, for every$x, y \in X$, and $a \in \mathbb{R}$, it satisfies the following conditions (Kreyszig, 1978):

i. Nonnegativity: $x \geq 0$,
ii. $\|x\| = 0$ if and only if $x = \hat{0}$ (the zero element of X),
iii. $\|ax\| = |a|\|x\|$,

iv. Triangle inequality: $\|x+y\| \leq \|x\| + \|y\|$.

The following definition of limit is the generalised version of that of real number space whose metric is the Euclidean metric.

Definition 2.3 (Limit in Metric Space). Let $f: A \rightarrow Y$ be a function from $A \subseteq X$ for a metric space (X, d_x) to a metric space (Y, d_Y). The limit of x approaching c of $f(x)$ equals L, or $\lim_{x \to c} f(x) = L$, if for every positive ε, there exists a positive δ such that $0 < d_X(x, c) < \delta$ implies $d_Y(f(x), L) < \varepsilon$ (Rudin, 1986).

One may construct a definition of derivative in metric space using the same generalisation of developing Definition 2.3. However, the idea of derivative was originated from the concept of slopes of tangents in real-valued functions (Bartle & Sherbert, 2011). Consequently, this visual-based concept cannot be simply generalised as that of limits since not every metric space has straightforward visualisation as up-to-three-dimensional real number spaces.

Despite the visualisation limitation, there are several attempts to define derivatives in metric spaces (and normed spaces). In this paper, we examine such three definitions from (Charatonik & Insall, 2012), (Ambrosio, Gigli, & Savaré, 2005), and (Coleman, 2012) respectively.

Definition 2.4 (Absolute Derivative). Let (X, d_X) and (Y, d_Y) be metric spaces and $f: A \rightarrow Y$ be a function where $A \subseteq X$. The function f is differentiable at point $c \in A$ if

$$\lim_{x \to c} \frac{d_Y\big(f(x), f(c)\big)}{d_X(x, c)}$$

exists. The limit is called the derivative of $f(x)$ at c denoted by $f'(c)$.

According to the definition of standard derivative (Bartle & Sherbert, 2011), we can denote the distance between x and c by h. Therefore, if the Definition 2.4 asserts

$$f'(c) = \lim_{x \to c} \frac{d_Y\big(f(x), f(c)\big)}{d_X(x, c)}$$

i.e. the derivative at point c is when x approaches c, we then can modify it as h approaching 0. In other words, we have

$$f'(x) = \lim_{h \to 0} \frac{d_Y\big(f(x+h), f(x)\big)}{d_X(h, 0)}.$$

Definition 2.5 (Parameterised Derivative). Let $f: (a, b) \to X$ be a function from an open interval $(a, b) \subseteq \mathbb{R}$ to a metric space (X, d). The function f is differentiable at x if

$$\lim_{h \to 0} \frac{d\big(f(x+h), f(x)\big)}{|h|}$$

exists. The derivative of $f(x)$ with respect to x is the limit.

Parameterised derivative is not the official name of the derivative. However, we need to name it in order to distinguish it among the other two derivatives. The name is based on the fact that the derivative is based on parameterised paths on a metric space (Ambrosio, Gigli, & Savaré, 2005).

Definition 2.6 (Gateaux Derivative). Let $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ be normed spaces and $f: A \to Y$ be a function where $A \subseteq X$. The function f is Gateaux differentiable at point $c \in A$ if for a nonzero element u of X,

$$\lim_{x \to 0} \frac{f(c + xu) - f(c)}{x}$$

exists. The limit is called the derivative of $f(x)$ at c denoted by $f'(c)$.

Even though these derivative definitions are different, we notice the similarities between the definitions. The limits of these definitions are similar to that of the definition of the standard derivative of real-valued functions indicating that the derivatives use the same approach. These limits illustrate a function value approaching another one as its input approaches that of the other function value.

2.2 Newton-Raphson Method

The Newton-Raphson is an iterative numerical method to find a root of a real-valued function. Let $f: A \to \mathbb{R}$ be a continuous real function. The root of f is approximated by the iteration

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

with a taken x_0 as the initial value. The iteration converges to the root if certain conditions are satisfied (Epperson, 2013).

The iteration certainly involves a derivative, and it is the standard one of real-valued functions. In this case, we discover the Newton-Raphson method of real-valued functions in different one-dimensional real metric spaces using the three derivative definitions. The developed Newton-Raphson iterations use different "distances" used in the derivative for each metric derivative.

2.3 Derivative of Real-Valued Functions in Discrete Metric Space

Discrete metric space is a metric space whose metric results either 0 if the elements are the same or 1 if these elements are different (Kreyszig, 1978).

Lemma 2.1. Let $(\mathbb{R}, d)$ be a space where $d: \mathbb{R}^2 \to \mathbb{R}$ is a function defined by

$$d = \begin{cases} 1, x \neq y \\ 0, x = y \end{cases}.$$

for $x, y \in \mathbb{R}$. Then, the space $(\mathbb{R}, d)$ is a metric space called the discrete metric space.

We can construct derivatives of real-valued functions with the three definitions except for the Gateaux derivative since the discrete metric space cannot be induced into a normed space as the following Lemma 2.2 proves.

Lemma 2.2. A discrete metric space cannot be induced by any normed space.

Proof. Suppose that a discrete metric space (X, d) could be induced by a normed space $(A, \|.\|)$. Then, by (Kreyszig, 1978), for every $x, y \in X$,

$$d(x, y) = \|x - y\|.$$

Hence, we have

$$\|x-y\| = \begin{cases} 0, x = y \\ 1, x \neq y \end{cases}.$$

Since $(A, \|.\|)$ is a normed space, it satisfies the properties of norms as defined by Definition 2.2. Therefore, it must be that for a non-zero-and-non-unit scalar α of A,

$$|\alpha|\|(x-y)\| = \|\alpha(x-y)\|.$$

Suppose that $x \neq y$ so that $x \neq \alpha y$ and $\|x-y\| = 1$. Then, we have

$$|\alpha|\|x-y\| = |\alpha|\|x-y\| = |\alpha|$$

and

$$\|\alpha(x-y)\| = \|\alpha x - \alpha y\| = 1$$

which lead to $|\alpha|\|(x-y)\| \neq \|\alpha(x-y)\|$, a contradiction and is only satisfied if $\alpha = 0$ or $\alpha = 1$. Therefore, any normed space cannot induce a discrete metric space. ■

Based on Lemma 2.2, we only construct the derivatives defined by Definition 2.4 (absolute derivative) and Definition 2.5 (parameterised derivative).

In this paper, we limit the scope of real-valued functions observed to injective and constant ones. The derivatives of functions other than these two need to be investigated further in future research.

Theorem 2.1. An injective function $f: A \subseteq \mathbb{R} \to \mathbb{R}$ is differentiable on an open interval $A \subseteq \mathbb{R}$ under the absolute derivative, and $f'(x) = 1$ for every $x \in A$.

Proof. Let ε be arbitrary positive number and $c \in A$. Choose $\delta = \varepsilon + 1$ that is indeed positive. Suppose that $x \neq c$ since $x = c$ implies that $0 < d(x,c) < \delta$ is untrue. Thus, $d(x,c) = 1$ and

$$0 < d(x,c) < 1 + \varepsilon \Leftrightarrow 0 < d(x,c) < \delta.$$

Since $x \neq c$ and f injective, we have $f(x) \neq f(c)$ for every $x, c \in A$. Therefore,

$$\left|\frac{d\big(f(x), f(c)\big)}{d(x,c)} - 1\right| = \left|\frac{1}{1} - 1\right| = |1-1| = 0 < \varepsilon.$$

By Definition 2.4, it is shown that f is differentiable on A under the absolute derivative, and $f'(c) = 1$ for every $c \in A$. ■

Theorem 2.2. A constant function $f: A \subseteq \mathbb{R} \to \mathbb{R}$ is differentiable on an open interval $A \subseteq \mathbb{R}$ under the absolute derivative, and $f'(x) = 0$ for every $x \in A$.

Proof: Let ε be arbitrary positive number and $c \in A$. Choose $\delta = \varepsilon + 1$ that is indeed positive Suppose that $x \neq c$ since $x = c$ implies that $0 < d(x,c) < \delta$ is untrue. Thus, $d(x,c) = 1$ and

$$0 < d(x,c) < 1 + \varepsilon \Leftrightarrow 0 < d(x,c) < \delta.$$

Since f constant, we have $f(x) = f(c)$ for every $x, c \in A$. Therefore,

$$\left|\frac{d\big(f(x), f(c)\big)}{d(x,c)} - 0\right| = \left|\frac{0}{1} - 0\right| = |0 - 0| = 0 < \varepsilon.$$

By Definition 2.4, it is shown that f is differentiable on A under the absolute derivative, and $f'(c) = 0$ for every $c \in A$. ■

Example 2.1. Let $f: A \subseteq \mathbb{R} \to \mathbb{R}$ with $f(x) = \sin x$ for every $x \in A$. Since f is injective on $A = \left(0, \frac{\pi}{2}\right)$, f is differentiable on A under the absolute derivative with $f'(x) = 1$ for every $x \in A$ by Theorem 2.1. ■

Example 2.2. Let $f: \mathbb{R} \to \mathbb{R}$ with $f(x) = x + 1$ for every $x \in \mathbb{R}$. Hence, f is injective on $\mathbb{R}$. By Theorem 2.1, f is differentiable everywhere with $f'(x) = 1$ for every $x \in \mathbb{R}$. We next to show how to use the $\varepsilon - \delta$ definition of the absolute derivative to prove that the function f is differentiable at $x = 2$ under the absolute derivative with $f'(2) = 1$. For every $\varepsilon > 0$, choose a positive number $\delta = 1 + \varepsilon$. Thus, if $0 < d(x,2) < \delta$, satisfied by $x \neq 2$, then

$$\left|\frac{d\big(f(x), f(2)\big)}{d(x,2)} - 1\right| = \left|\frac{d(x+1, 2+1)}{d(x,2)} - 1\right| = \left|\frac{d(x+1, 3)}{d(x,2)} - 1\right|$$

$$= \left|\frac{1}{1} - 1\right| = |1 - 1| = 0 < \varepsilon.$$ ■

Now, consider the parameterised derivative.

Theorem 2.3. A constant function $f: A \subseteq \mathbb{R} \to \mathbb{R}$ is differentiable on an open interval $A \subseteq \mathbb{R}$ under the parameterised derivative, and $f'(x) = 0$ for every $x \in A$.

Proof. We show that $f'(x) = 0$. Let ε be an arbitrary positive number. Choose $\delta = 1 + \varepsilon$ that is indeed positive. Let h be a real number such that $0 < d(h, 0) < \delta$. Since $\delta = 1 + \varepsilon > 1$, it must be that $h \neq 0$, and since f is constant, we have $f(x) = f(x + h)$ for every $x \in A$ and $h \in \mathbb{R}$. Therefore,

$$\left|\frac{d\big(f(x+h), f(x)\big)}{|h|} - 0\right| = \left|\frac{d(k,k)}{|h|} - 0\right| = |0 - 0| = 0 < \varepsilon. \qquad \blacksquare$$

Theorem 2.4. An injective function $f: A \subseteq \mathbb{R} \to \mathbb{R}$ is not differentiable on an open interval $A \subseteq \mathbb{R}$ under the parameterised derivative.

Proof. To prove that f is not differentiable at A, we use the negation of Definition 2.5. In other words, we show that for every $L \in \mathbb{R}$, there exist $\varepsilon > 0$ and $x \in A$ implying that for every $\delta > 0$, there exists $h \in \mathbb{R}$ and $x \in A$ such that $0 < d(h, 0) < \delta$ but $\left|\frac{d(f(x+h), f(x))}{|h|} - L\right| \geq \varepsilon$. We consider two cases: $L = 0$ and $L \neq 0$.

Suppose that $L = 0$. Choose $\varepsilon = \frac{1}{2} > 0$, and let δ be arbitrary positive real number. Take $h \in \mathbb{R}$ such that $0 < |h| \leq 1$ and $x \in (a, b)$. Then, it must be that $h \neq 0$, $\frac{1}{|h|} \geq 1$, and $d\big(f(x+h), f(x)\big) = 1$. For $\delta < 1$, the statement $0 < d(h, 0) < \delta$ is untrue since $h \neq 0$ implies $d(h, 0) = 1$. Therefore, the statement "if $0 < d(h, 0) < \delta$, then $\left|\frac{d(f(x+h), f(x))}{|h|} - L\right| \geq \varepsilon$" is true by vacuous proof. For $\delta \geq 1$, it satisfies $0 < d(h, 0) < \delta$. Therefore, we have

$$\left|\frac{d\big(f(x+h), f(x)\big)}{|h|} - L\right| = \left|\frac{1}{|h|} - 0\right| = \frac{1}{|h|} \geq 1 > \frac{1}{2} = \varepsilon.$$

Thus if δ is arbitrary positive, we have

$$\left|\frac{d\big(f(x+h), f(x)\big)}{|h|} - L\right| > \varepsilon.$$

Suppose that $L \neq 0$. Choose $\varepsilon = \frac{|L|}{2}$ that is indeed positive. Let δ be arbitrary positive real number. Take $h \in \mathbb{R}$ such that $|h| \leq \frac{1}{|2L|}$ and $x \in (a, b)$. Therefore, it must be that $|h| \neq 0$, $\frac{1}{|h|} \geq |2L|$, and $d\big(f(x+h), f(x)\big) = 1$. For $\delta < 1$, the statement $0 < d(h, 0) < \delta$ is untrue since $h \neq 0$ implies $d(h, 0) = 1$. Therefore, the statement "if $0 < d(h, 0) < \delta$, then $\left|\frac{d(f(x+h), f(x))}{|h|} - L\right| \geq \varepsilon$" is true by vacuous proof. For $\delta \geq 1$, it satisfies $0 < d(h, 0) < \delta$. Then, by the triangle inequality, we have

$$\left|\frac{d\big(f(x+h), f(x)\big)}{|h|} - L\right| = \left|\frac{1}{|h|} - L\right| \geq \frac{1}{|h|} - |L| \geq |2L| - |L|$$

$$= 2|L| - |L| = |L| > \frac{|L|}{2} = \varepsilon. \qquad \blacksquare$$

Now we have both derivatives that result in constant functions. By Theorem 2.2 and Theorem 2.3, since the absolute and parameterised derivative of constant functions are 0, another constant function, the second derivative of both derivatives also 0 and so on. Therefore, we have the following result for the high-order derivatives.

Corollary 2.1. If f is a constant function of the discrete metric space, then $f^{(n)}(x) = 0$ under absolute and parameterised derivative for $= 1,2,3,4, \ldots$.

The proof of Corollary 2.1 is derived from Theorem 2.2 and 2.3 by induction.

2.4 Newton-Raphson Method in Discrete Metric Space

Let $f: A \to \mathbb{R}$ be a real-valued function where A is a subset of the real discrete metric space. We develop the Newton-Raphson methods to find a root of f using the iteration

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

and the constructed absolute and parameterised derivative. The $f'(x_n)$ in the iteration is adjusted to the derivatives.

Since constant functions do not have a root except for $f(x) = 0$, it is not necessary to construct the Newton-Raphson method for such functions. Thus, we recall the absolute derivative constructed by Theorem 2.1 for injective function. That is, for every differentiable injective function $f: A \to \mathbb{R}$,

$$f'(x) = 1.$$

Then, for $x = x_n$, we have

$$f'(x_n) = 1.$$

Assuming $f(x_n + h) \neq f(x_n)$, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{1} \Leftrightarrow x_{n+1} = x_n - f(x_n).$$

Depending on the choice of x_0, the iteration could converge to the root. Moreover, (Epperson, 2013) provides the following theorem that determines the convergence of the Newton-Raphson method for the standard derivative.

Theorem 2.5. Let $f: A \to \mathbb{R}$ be a real-valued function differentiable twice at x, and for some a, $f(a) = 0$ i.e. a is a root of f. Define

$$M = \frac{\max_{x \in A} |f''(x)|}{2 \min_{x \in A} |f'(x)|}$$

that is assumed to be not infinity. Then, for any initial value x_0 satisfying

$$M|a - x_0| < 1,$$

The Newton-Raphson iteration converges.

We examine if Theorem 2.1 also holds for the Newton-Raphson method for absolute derivative. Let $f: A \to \mathbb{R}$ be a real-valued injective function of the discrete metric space. Then, by Theorem 2.1 and Corollary 2.1, for every $x \in A$.

$$f'(x) = 1$$

and

$$f''(x) = 0.$$

Then, we have

$$\min_{x \in A} |f'(x)| = 1.$$

and

$$M = \frac{\max_{x \in A} |f''(x)|}{2 \min_{x \in A} |f'(x)|} = \frac{0}{2(1)} = 0.$$

Therefore, the expression

$$M|a - x_0| < 1$$

is always satisfied since $0 < 1$, thus implying that the Newton-Raphson iteration always converges regardless of the choice of x_0. This claim appears to be bombastic and definitely needs further investigation.

Another convergence criterion of the Newton-Raphson method is also examined in (Khandani, 2021). We suppose the notation $f(a+)$ and $f(b-)$ mean left-derivative and right-derivative respectively. The one-side direction is determined by the direction of limit in the derivative definition. Here is the criterion.

Theorem 2.6. Let $f: [a, b] \to \mathbb{R}$ be a real-valued function where b is the unique root of f in $[a, b]$, and f is differentiable twice on (a, b). If $f(x)f''(x) \geq 0$, $f(x)f'(x) < 0$, and $f'(x) \neq 0$ for every $x \in (a, b)$, $f'(a+) \neq 0$, and $f'(b-) \neq 0$[1], for an initial value $x_0 \in [a, b]$, then the Newton-Raphson iteration converges to b.

If $f: [a, b] \to \mathbb{R}$ is an injective function, $f'(x) = 1$ and $f''(x) = 0$ for every $x \in [a, b]$. Then, $f(x)f''(x) \geq 0$, $f'(x) \neq 0$, $f'(a+) \neq 0$, and $f'(b-) \neq 0$[2]. If $x \geq 0$, then $f(x)f'(x) < 0$ is untrue, and the theorem holds by vacuous proof. If $x < 0$, then $f(x)f'(x) < 0$ is true implying that the theorem holds.

Since the theorem holds for injective functions, the Newton-Raphson always converges. However, we should investigate it further since claiming that the iteration always converges requires a mathematically valid proof.

Here is an example of the use of the Newton-Raphson.

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Example 2.3. Find the root of the function $f: [-4.5, -2] \to \mathbb{R}$ defined by $f(x) = e^x - \sin x$ with the Newton-Raphson method for initial values -2.5 and -4.5.

Solution. According to the function graph illustrated by GeoGebra as in Figure 2.1, f has a root around -3.2. Here is the iteration x_i for $i = 0,1,2,\dots,11$ with $x_0 = -2.5$.

Table 2.1 The Iteration for $x_0 = -2.5$

i	0	1	2	3	4	5
x_i	0.5	$-3{,}18056$	$-3{,}18317$	$-3{,}18306$	$-3{,}18306$	$-3{,}18306$
i	6	7	8	9	10	11
x_i	$-3{,}18306$	$-3{,}18306$	$-3{,}18306$	$-3{,}18306$	$-3{,}18306$	$-3{,}18306$

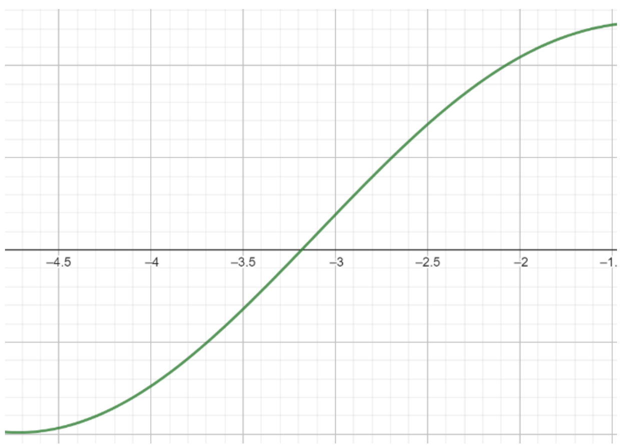

Figure 2.1 The Graph of $f(x) = e^x - \sin x$ at $[-4.5, -2]$

Here is such iteration for $x_0 = -4.5$.

Table 2.2 The Iteration for $x_0 = -4.5$

i	0	1	2	3	4	5
x_i	$-4{,}5$	$-3{,}53358$	$-3{,}18075$	$-3{,}18316$	$-3{,}18306$	$-3{,}18306$
i	6	7	8	9	10	11
x_i	$-3{,}18306$	$-3{,}18306$	$-3{,}18306$	$-3{,}18306$	$-3{,}18306$	$-3{,}18306$

It can be seen from Table 2.1 and Table 2.2 that the iterations seem to converge or approach the root despite the initial values of both relatively far from the actual root. Moreover, the convergence is fast; it can be seen even before the 10th iteration. ■

This one example (Example 2.3) alone does not represent the overall convergence of the Newton-Raphson scheme. However, it does indicate the possibility of finding a root using the scheme. Further investigation is interesting for future research.

The only functions differentiable under parameterised derivative are constant functions, and constant functions do not have a root unless it equals 0. Therefore, it is not necessary to construct the Newton-Raphson method for this derivative.

3. CONCLUSION AND SUGGESTION

Not every function in one-dimensional real discrete metric space whose domain is a subset of the set of all real numbers is differentiable under absolute derivative and parameterised derivative. Examples of functions that are differentiable under absolute derivative are constant functions and injective functions whose derivative are 0 and 1 respectively, and the functions that are differentiable under parameterised derivative are constant functions whose derivative is 0. The definability of absolute derivatives for injective functions enables the construction of the Newton-Raphson methods for the corresponding derivative, and the iterations indicate the possibility of a new approach to function root approximation.

This research still has gaps that have potential to become further research topics. They include deeper investigation of the derivative properties, such as linearity and patterns of certain functions, and such derivatives for non-constant and non-injective functions. Other interesting ones are convergence criteria of the Newton-Raphson method in discrete metric space.

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