

## FULLY GENERALIZED MODULE

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**ABSTRACT.** *This paper introduces the concept of fully generalized modules as an extension of classical module theory over generalized algebraic structures. We prove that every fully generalized module forms a normal generalized group and contains abelian subsets. We further establish the existence of a trivial fully generalized submodule and show that the family of fully generalized submodules is closed under sums and intersections. In addition, we prove the existence of fully generalized module homomorphisms and derive their elementary properties. These results extend existing work on generalized modules and provide a broader theoretical framework for studying module-like structures in generalized algebra. Potential directions for future research include the study of direct sums, fully generalized torsion modules, and fully generalized vector spaces.*

**Keywords:** *Generalized group, generalized ring, fully generalized module, submodule, homomorphism.*

**ABSTRAK.** Penelitian ini memperkenalkan konsep modul tergeneralisasi penuh sebagai perluasan dari teori modul klasik dalam kerangka struktur aljabar tergeneralisasi. Dibuktikan bahwa setiap modul tergeneralisasi penuh membentuk grup tergeneralisasi normal dan memuat subset yang merupakan grup abelian. Selain itu, ditunjukkan keberadaan submodul trivial serta dibuktikan bahwa himpunan submodul tergeneralisasi penuh tertutup terhadap operasi penjumlahan dan irisan. Penelitian ini juga membuktikan keberadaan homomorfisma modul tergeneralisasi penuh dan mengkaji sifat-sifat elementernya. Hasil ini memperluas teori modul tergeneralisasi dan menyediakan kerangka teoretis yang lebih luas dalam kajian struktur modul pada konteks tergeneralisasi. Arah penelitian selanjutnya mencakup kajian jumlah langsung, modul torsi tergeneralisasi penuh, serta ruang vektor tergeneralisasi.

**Kata Kunci:** Grup tergeneralisasi, ring tergeneralisasi, modul tergeneralisasi penuh, submodul, homomorfisma.

## 1. INTRODUCTION

Generalized groups constitute an extension of classical group theory in which the identity axiom is modified so that each element possesses its own identity element. This concept was introduced by (Fatehi & Molaei, 2012) and has since been developed in several directions, including elementary properties (J.O Adéníran et al., 2010), finite generalized groups (Poursalavati, 2024), generalized semigroups (Fatehi & Molaei, 2012), generalized actions (Maleki & R. Molaei, 2024) generalized rings (Fatehi & Molaei, 2017), elementary properties of

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generalized rings (Rosyadi, 2024), prime generalized rings (Gorton et al., 2016) and their applications (Jaiyeola & Smarandache, 2018).

Meanwhile, modules are generalizations of vector spaces, classically defined as abelian groups equipped with scalar multiplication over a ring with unity. (G. Romeo & Sneha, 2020) extended the concept of modules by replacing the underlying abelian group with a generalized group, and further developments led to the notion of generalized cross modules (Fulya Akiz, 2024).

Motivated by these developments, a natural question arises as to whether both the ring and the underlying group in the classical module structure can be replaced simultaneously by generalized structures. While existing researches have explored variations of generalized modules, such as generalized modules over generalized rings via module groupoids by Romeo and Sneha (2020). They often restrict certain operational behaviors or focus primarily on category-theoretic properties. This leads to the introduction of the concept of fully generalized modules. The advantage of the topic discussed in this article compared with the existing researches lies in its axiomatic generalization where the identity map operates locally on both structures without losing the compatibility of scalar multiplication, thereby providing a more comprehensive and symmetric framework.

The aim of this study is to define and further develop the theory of fully generalized modules and to investigate their fundamental algebraic properties, including submodule structures, homomorphisms, and closure properties under sums and intersections. The main contribution of this work lies in extending the framework of generalized algebraic structures by providing a new theoretical foundation for module theory in a fully generalized setting.

## 2. THEORETICAL FOUNDATION

(G. Romeo & Sneha, 2020) introduced the concept of generalized modules, where a generalized module is defined as a generalized group satisfying the axioms of scalar multiplication over a ring with unity. The definition of a generalized module is given below.

**Definition 2.1** (G. Romeo & Sneha, 2020) Let  $R$  be a ring with unity  $1_R$ . A generalized group  $M$  is said to be a (left) generalized  $R$ -module if there exist an action  $R \times M \rightarrow M$  (denoted by  $(r, \alpha) \mapsto r\alpha$ ) such that for all  $r, s \in R$  and  $\alpha, \beta \in M$ , the following conditions hold:

1.  $(r + s)\alpha = r\alpha + s\alpha$
2.  $r(\alpha + \beta) = r\alpha + r\beta$
3.  $r(s\alpha) = (rs)\alpha$
4.  $re(\alpha) = e(\alpha)$ , where  $e(\alpha) \in M$  is the identity element of  $\alpha$
5.  $1_R\alpha = \alpha$

### 3. RESULTS AND DISCUSSION

This section presents the definition of fully generalized modules and discusses their fundamental properties. In addition, it provides an overview of the basic structural characteristics that arise within this framework.

#### 3.1 Fully Generalized Module

A generalized module, as introduced by (G. Romeo & Sneha, 2020), is a generalized group endowed with a scalar multiplication satisfying the module axioms over a ring with unity. If the scalar multiplication is defined over a generalized ring and satisfies the corresponding axioms, then the structure is referred to as a fully generalized module.

**Definition 3.1.1.** Let  $(R, +_R, \cdot_R)$  be a generalized ring with unity  $1_R$ , and let  $(M, +_M)$  be a generalized group. A scalar multiplication on  $M$  over  $R$  is a mapping from  $R \times M$  to  $M$ , denoted by  $(r, \alpha) \rightarrow r\alpha$ . The structure  $M$  is called a fully generalized module over  $R$  if for all  $r, s \in R$  and  $\alpha, \beta \in M$ , the following conditions are satisfied:

1.  $r(\alpha +_M \beta) = r\alpha +_M r\beta$
2.  $(r +_R s)\alpha = r\alpha +_M s\alpha$
3.  $r(s\alpha) = (r \cdot_R s)\alpha$
4.  $re_M(\alpha) = e_M(r\alpha)$  where  $e_M(\alpha) \in M$  is the identity element of  $\alpha$
5.  $1_R\alpha = \alpha$

Here the example of fully generalized module.

**Example 3.1.2.** Let  $M = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in R \right\}$  where  $R$  is a generalized ring with unity 1. Define the addition on  $M$  and scalar multiplication by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a & f \\ c & d+h \end{pmatrix},$$

$$r \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & rd \end{pmatrix},$$

for  $r \in R$  and  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in M$ . Then  $M$  is fully generalized module.

Theorem 3.1.3 presents fundamental properties related to inverse elements and identity elements in fully generalized modules.

**Theorem 3.1.3.** Let  $M$  be a fully generalized module over a generalized ring  $R$  with unity. If  $\alpha \in M$  and  $r \in R$ , then the following properties hold:

1.  $-(r\alpha) = r(-\alpha)$
2.  $e_R(r)\alpha = e_M(r\alpha)$  where  $e_M(\alpha) \in M$  is identity element of  $\alpha$  and  $e_R(r) \in R$  is the identity element of  $r$ .

**Proof.** By Definition 3.1.1 and properties of generalized structures, we have  $r\alpha + r(-\alpha) = r(\alpha + (-\alpha)) = re_M(\alpha) = e_M(r\alpha)$  and  $r(-\alpha) + r\alpha = e_M(r\alpha)$ , which implies  $-(r\alpha) = r(-\alpha)$  by the uniqueness of the local inverse. Moreover, the relations  $r\alpha + e_R(r)\alpha = (r + e_R(r))\alpha = r\alpha = (e_R(r) + r)\alpha = e_R(r)\alpha + r\alpha$ . Directly yield  $e_R(r)\alpha = e_M(r\alpha)$  due to the uniqueness of the local identity element  $e_M(r\alpha)$ . ■

Theorem 3.1.4 establishes that every fully generalized module is a normal generalized group in the sense of Definition 2.4 (J.O Adéníran et al., 2010).

**Theorem 3.1.4.** Let  $M$  be a fully generalized module. Then  $M$  is a normal generalized group.

**Proof.** For any  $\alpha, \beta \in M$ , by definition 3.1.1 and theorem 3.1.3, we have  $e_M(\alpha + \beta) = 1_R e_M(\alpha + \beta) = e_R(1_R)(\alpha + \beta) = e_R(1_R)\alpha + e_R(1_R)\beta = 1_R e_M(\alpha) + 1_R e_M(\beta) = e_M(\alpha) + e_M(\beta)$ , where  $1_R$  is the unity of  $R$ , while  $e_R$  and

$e_M$  denote the local identities in  $R$  and  $M$  respectively. This homomorphic property of the local identity map ensures that  $M$  is a normal generalised group. ■

The following theorem shows that if a generalized ring  $R$  is invertible, then the identity scalar multiplication is a morphism.

**Theorem 3.1.5.** Let  $M$  be a fully generalized module and let  $R$  be a generalized ring with unity. If  $R$  is invertible for some  $r \in R$ , then  $e_R(s)\alpha = e_R(s)e_M(\alpha)$  for all  $s \in R$  and  $\alpha \in M$ .

**Proof.** From Theorem 2.11 (Fatehi & Molaei, 2017) we obtain  $e_R(s) = s$  for all  $s \in R$ . Then  $e_R(s)\alpha = se_M(\alpha) = e_R(s)e_M(\alpha)$ . ■

Within a fully generalized module, it is possible to construct a subset that constitutes an abelian group.

**Theorem 3.1.6.** Let  $M$  be a fully generalized module. Then the subset  $M_a = \{\alpha \in M | e_M(\alpha) = e_M(a)\}$  is an abelian group.

**Proof.** It is straightforward to verify that  $M_a$  is a group. To prove commutativity axiom, we have  $(1 + 1)(\alpha + \beta) = \alpha + \alpha + \beta + \beta$  and  $(1 + 1)(\alpha + \beta) = \alpha + \beta + \alpha + \beta$ . Thus  $\alpha + \alpha + \beta + \beta = \alpha + \beta + \alpha + \beta$ . Adding  $-\alpha$  to both sides yields  $e_M(\alpha) + \alpha + \beta + \beta = e_M(\alpha) + \beta + \alpha + \beta$  furthermore adding  $-\beta$  to both sides gives  $\alpha + \beta + e_M(\beta) = e_M(\alpha) + \beta + \alpha + e_M(\beta)$ . Since  $\alpha, \beta \in M_a$ , we have  $e_M(\alpha) = e_M(\beta) = e_M(a)$ . Therefore  $\alpha + \beta + e_M(a) = e_M(a) + \beta + \alpha$  which implies  $\alpha + \beta = \beta + \alpha$ . Then  $M_a$  is an abelian group. ■

The next topic discussed in this paper is the concept of fully generalized submodules. In this study, we establish the existence of the sum of fully generalized submodules, the intersection of generalized submodules, and the trivial fully generalized submodule.

### 3.2 Fully Generalized Submodule

In this section, we extend the concept of sub-structures to fully generalized modules. Naturally, a submodule can be viewed as a nonempty subset that inherits the underlying algebraic structure of the parent module under the restricted operations, as defined below.

**Definition 3.2.1.** Let  $M$  be a fully generalized module over a generalized ring  $R$  with unity. A nonempty subset  $N \subseteq M$  is called a fully generalized submodule of  $M$  if  $N$  is a generalized subgroup of  $M$  and is closed under scalar multiplication; that is, for every  $r \in R$  and every  $\alpha \in N$ , the product  $r\alpha$  belongs to  $N$ .

Building upon the construction in Example 3.1.2, we present a further instance of a fully generalized submodule within a matrix module endowed with non-standard operations.

**Example 3.2.2.** Given  $M = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in R \right\}$  is a fully generalized module over generalized ring  $R$  with unity 1 and  $N = \left\{ \begin{pmatrix} a & b \\ a & c \end{pmatrix} \mid a, b, c \in R \right\}$ . Define the addition on  $M$  and scalar multiplication by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a & f \\ c & d+h \end{pmatrix},$$

$$r \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & rd \end{pmatrix},$$

for  $r \in R$  and  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in M$ . Then  $N$  is fully generalized submodule.

A preliminary observation is that each fully generalized module  $M$  always admits at least two distinguished fully generalized submodules:  $M$  itself and the zero submodule  $E = \{e_M(\alpha) \mid \alpha \in M\}$ . The latter is often called the trivial submodule.

**Theorem 3.2.3.** Let  $M$  be a fully generalized module over a generalized ring  $R$  with unity. Let  $e_M : M \rightarrow M$  be the generalized identity map, and define the subset  $E = \{e_M(\alpha) \mid \alpha \in M\}$ . Then  $E$  is a fully generalized submodule of  $M$  over  $R$ .

**Proof.** Since  $M$  is nonempty, there exists  $\alpha_0 \in M$ . By definition,  $e_M(\alpha_0) \in E$ , hence  $E \neq \emptyset$ . Let  $\xi, \mu \in E$ . Then there exist  $\alpha, \beta \in M$  such that  $\xi = e_M(\alpha)$  and  $\mu = e_M(\beta)$ . Because  $e_M$  is additive, we have  $\xi - \mu = e_M(\alpha) - e_M(\beta) = e_M(\alpha - \beta)$ . Since  $\alpha - \beta \in M$ , it follows that  $\xi - \mu \in E$ . Thus,  $E$  is a generalized subgroup of  $M$ . Next, let  $\xi \in E$  and  $r \in R$ . Then  $\xi = e_M(\alpha)$  for some  $\alpha \in M$ . By homogeneity of  $e_M$ ,  $r\xi = re_M(\alpha) = e_M(r\alpha)$ . Since  $r\alpha \in M$ , we conclude that  $r\xi \in E$ .

Therefore,  $E$  is nonempty, closed under subtraction, and closed under scalar multiplication, and hence  $E$  is a fully generalized submodule of  $M$ . ■

If we have a fully generalized module, and two fully generalized submodule, then the addition of two fully generalized submodule is a fully generalized submodule.

**Theorem 3.2.4.** Let  $M$  be a fully generalized module with commutative addition. If  $K, L \subseteq M$  are fully generalized submodules, then their sum  $K + L = \{k + l \mid k \in K, l \in L\}$  is a fully generalized submodule of  $M$ .

**Proof.** Since  $K$  and  $L$  are nonempty, there exist  $\kappa \in K$  and  $\rho \in L$ , implying  $\kappa + \rho \in K + L$ . Hence,  $K + L \neq \emptyset$ . Let  $\xi = \kappa_1 + \rho_1$  and  $\mu = \kappa_2 + \rho_2$  be elements of  $K + L$ , where  $\kappa_1, \kappa_2 \in K$  and  $\rho_1, \rho_2 \in L$ . Then  $\xi + \mu = (\kappa_1 + \kappa_2) + (\rho_1 + \rho_2) \in K + L$  and  $\xi - \mu = (\kappa_1 - \kappa_2) + (\rho_1 - \rho_2) \in K + L$  since  $K$  and  $L$  are closed under addition and subtraction. For scalar multiplication, let  $r \in R$ . Then  $r(\kappa + \rho) = r\kappa + r\rho \in K + L$ , because  $r\kappa \in K$  and  $r\rho \in L$ . Thus,  $K + L$  is nonempty, closed under subtraction, and closed under scalar multiplication, and therefore is a fully generalized submodule of  $M$ . ■

Next if two fully generalized modules have nonempty intersection, then the intersection is a fully generalized submodule.

**Theorem 3.2.5.** Let  $M$  be a fully generalized module. If  $K, L \subseteq M$  are fully generalized submodules and  $K \cap L \neq \emptyset$ , then the intersection  $K \cap L = \{\xi \mid \xi \in K \text{ and } \xi \in L\}$  is a fully generalized submodule of  $M$ .

**Proof.** Let  $\xi, \mu \in K \cap L$ . Then  $\xi, \mu \in K$  and  $\xi, \mu \in L$ . Since both  $K$  and  $L$  are fully generalized submodules, we have  $\xi - \mu \in K$  and  $\xi - \mu \in L$ , which implies  $\xi - \mu \in K \cap L$ . Now let  $r \in R$ . Because  $\xi \in K \cap L$ , we have  $\xi \in K$  and  $\xi \in L$ , hence  $r\xi \in K$  and  $r\xi \in L$ , so  $r\xi \in K \cap L$ . Thus,  $K \cap L$  is closed under subtraction and scalar multiplication. Since it is assumed nonempty,  $K \cap L$  is a fully generalized submodule of  $M$ . ■

The final topic discussed in this paper concerns morphisms. In this section, we introduce the notion of fully generalized module homomorphisms and investigate their fundamental properties.

### 3.3 Fully Generalized Module Homomorphism

The following definition introduces fully generalized module homomorphisms as mappings that preserve the underlying module operations.

**Definition 3.3.1.** Let  $M$  and  $N$  be fully generalized modules over a generalized ring  $R$ . A map  $f: M \rightarrow N$  is called a fully generalized module homomorphism if it preserves the module operations; that is, for all  $a, b \in M$  and all  $r \in R$ , the following two conditions hold:

$$\begin{aligned} f(a + b) &= f(a) + f(b) \\ rf(a) &= f(ra) \end{aligned}$$

This is the example of fully generalized module homomorphism.

**Example 3.3.2.** Let  $M = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in R \right\}$  be a fully generalized module over a generalized ring  $R$  with unity 1. Define addition on  $M$  and scalar multiplication by

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} &= \begin{pmatrix} a & f \\ c & d + h \end{pmatrix}, \\ r \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} a & b \\ c & rd \end{pmatrix}, \end{aligned}$$

for  $r \in R$  and  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in M$ . The function  $f: M \rightarrow M$  defined

$$f \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & b \\ a & d \end{pmatrix}$$

is a fully generalized module homomorphism.

The next theorem discusses the elementary properties of fully generalized module homomorphism.

**Theorem 3.3.3.** Let  $f: M \rightarrow N$  be a fully generalized module homomorphism. Then for all  $x \in M$ , the following properties hold:

1.  $f(e(\xi)) = e(f(\xi))$

$$2. -f(\xi) = f(-\xi)$$

$$3. -f(-\xi) = f(\xi)$$

**Proof.** Since  $f$  is a fully generalized module homomorphism, the identity element of  $f(\xi)$  is  $e(f(\xi))$ . Using additivity, we obtain  $f(\xi) + f(e(\xi)) = f(\xi + e(\xi)) = f(\xi) = f(e(\xi) + \xi) = f(e(\xi)) + f(\xi)$  which implies  $f(e(\xi)) = e(f(\xi))$ . Next, consider additive inverses. Since  $f(\xi) - f(\xi) = e(f(\xi)) = f(e(\xi)) = f(e(-\xi)) = e(f(-\xi)) = f(-\xi) - f(-\xi)$ , it follows that  $f(-\xi) = -f(\xi)$ . Finally, the inverse of  $f(\xi)$  is  $-f(\xi)$ , and the inverse of  $f(-\xi)$  is  $-f(-\xi)$ . Since  $f(-\xi) = -f(\xi)$ , we have  $e(f(-\xi)) = e(-f(\xi)) = e(f(\xi))$ . By the uniqueness of identity elements, we conclude  $-f(-\xi) = f(\xi)$ . ■

The following theorem shows that for every fully generalized module, there always exists a fully generalized module homomorphism.

**Theorem 3.3.4.** Let  $M$  be a fully generalized module over a generalized ring  $R$  with unity. Then there exists a fully generalized module homomorphism  $f: M \rightarrow M$  defined by  $f(\xi) = e_M(\xi)$ , for all  $\xi \in M$ , where  $e_M: M \rightarrow M$  is the generalized identity map.

**Proof.** Let  $\xi, \mu \in M$  and  $r \in R$ . Since  $e_M$  is a generalized identity map, it preserves addition and scalar multiplication. Hence,  $f(\xi + \mu) = e_M(\xi + \mu) = e_M(\xi) + e_M(\mu) = f(\xi) + f(\mu)$  and  $f(r\xi) = e_M(r\xi) = re_M(\xi) = rf(\xi)$ . Therefore, the map  $f$  preserves both the module addition and scalar multiplication. By Definition 3.3.1,  $f$  is a fully generalized module homomorphism. ■

#### 4. CONCLUSSIONS AND SUGGESTIONS

The results of this study show that, in addition to preserving the fundamental properties of classical modules, a fully generalized module exhibits several important structural characteristics. In particular, we establish that a fully generalized module forms a normal generalized group. Furthermore, we prove that it contains subsets that constitute abelian groups, admits a trivial fully generalized submodule, and is closed under both the intersection and the sum of fully

generalized submodules. We also demonstrate that a fully generalized module homomorphism can always be constructed.

For future research, it is recommended to investigate direct sums in the context of fully generalized modules, to study fully generalized torsion modules, and to explore the structure of fully generalized vector spaces.

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