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DYNAMICS OF MULTIPLICATION Γ-SEMIGROUP

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ABSTRACT. *This paper gives details on the development of multiplication* Γ-*semigroups and presents a survey of existing results obtained in the literature. Substantial number of observations due to multiplication semigroups and multiplication* Γ-*semigroups, and research problems dealing with multiplication Γ-semigroups are also included.*

Keywords:*Multiplication ring, multiplication semigroup, multiplication* Γ-*semigroup.*

ABSTRAK. Artikel ini menjelaskan secara rinci perkembangan Γ-semigroup multiplikasi dan menyajikan survey hasil-hasil yang telah ada yang diperoleh dari literatur. Sejumlah penelaahan substansial mengenai semigrup multiplikasi dan Γ-semigroup multiplikasi, dan masalah-masalah penelitian yang berkaitan dengan Γ-semigroup multiplikasi juga disajikan dalam artikel ini.

Kata Kunci: Ring Multiplikasi, Semigrup Multiplikasi, Γ-semigroup Multiplikasi.

1. INTRODUCTION

The concept of multiplication rings was introduced to create an algebraic structure that incorporates the notion of ideal factorisation. Specifically, a ring R is termed a left (right) multiplication ring if, for any two ideals I and J of R where I is a subset of J, there exists an ideal K of R such that $I = KJ$ $(I = JK)$. A ring R is designated as a multiplication ring if it satisfies both left and right multiplication ring conditions. In the case of a commutative ring, R qualifies as a multiplication ring if and only if the set of ideals $(\mathcal{J}(R))$ is a subset of the set of R-modules $(Mod_m(R))$.

(Krull, 1948) originally introduced multiplication commutative ring as a generalization of Dedekind domain. The study of multiplication rings has garnered considerable attention from various researcher, leading to the application

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of potent ideal-theoretic techniques to characterise them extensively, as cited in (Gilmer & Mott, 1965), (Griffin, 1974), (Larsen & McCarthy, 1971), (Mori, 1934), (Mott, 1964), and (Mott, 1969).

With the advancement in semigroup theory, a parallel line of research emerged, introducing the concept of multiplication semigroups. (Mannepalli, 1975) pioneered this notion by utilising commutative semigroups, which are not groups, aiming to establish a framework akin to multiplication rings within the realm of semigroup theory. Subsequently, (Mannepalli & Satyanarayana, 1976) collaborated and studied the dual of multiplication semigroups in the non-communicative case.

Generalisation plays a very vital role in mathematics. In semigroups, Γ-semigroups are one of their generalisations. (Sen, 1981) initiated the concept of Γ-semigroups. Later, (Sen & Saha, 1986) revisited the definition of Γ-semigroup by pragmatically loosening the defining conditions of Γ-semigroup for expediency. The development of Γ-semigroups depends on the fact that subsets of a semigroup inherently inherit associativity but are not necessarily closed. As a result of this, various generalisations and similarities of corresponding results in semigroup theory have been obtained based on the modified definition in (Awolola & Ibrahim, 2024), (Dutta & Chattopadhyay, 2006), (Saha, 1987), (Sen & Saha, 1990), (Sen & Chattopadhyay, 2004), and (Seth, 1992).

In an attempt to broaden the theoretical aspect of Γ-semigroup theory, (Awolola & Ibrahim, 2023) introduced and established some properties of multiplication Γ-semigroup. The motivation behind their study is based on the two-fold concept of multiplication semigroups presented by (Mannepalli & Satyanarayana, 1976). The work of (Sen and Chattopadhyay, 2016) and (Ibrahim & Awolola, 2023) serves as motivation for this paper in that the research articles dealt extensively with a survey on Γ-semigroups without Γ-semigroups satisfying Γ-multiplication property.

2. PRELIMINARIES

We review some of the fundamental terminology in the theory of semigroups and Γ-semigroups from (Clifford and Preston, 1961), (Clifford and Preston, 1967), and (Sen and Saha, 1986), respectively.

Subsequent sections of this paper are organised according to commutative multiplication semigroups, non-commutative multiplication semigroups, multiplication Γ-semigroups in a non-commutative case, and research problems. The definitions and some existing results related to the aforesaid algebraic structures are reviewed in the following sections.

2.1 Semigroups

Let S be an arbitrary semigroup. If S has an identity, set $S^1 = S$. If S does not have an identity, let S^1 be the semigroup S with an identity adjoined, usually denoted by the symbol 1.

A subset A of S is a left (right) ideal of S if for every $a \in A$ and $s \in S$ we have $sa \in A$ ($as \in A$). If A is both left and right ideal, then it is an ideal of S. A (left, right) ideal A is an idempotent (left, right) ideal if $A = A^2$.

By a proper ideal of S, we mean an ideal distinct from S. The terms "properly contained in" and "contained in or equal" are represented by the symbols $<$ and $\subseteq$, respectively. For any (left, right) ideal A of S, $\bigcap_{n=1}^{\infty} A^n$ is denoted by A^w. A proper left (right) ideal A of S is a maximal left (right) ideal of S if A is not contained in any larger proper left (right) ideal of S. A semigroup S is called a left (right) simple if it contains no proper left (right) ideal, that is, for every $a \in S$, $Sa = S$ ($aS = S$). The semigroup S is called simple if it has no proper ideals. A semigroup S is semisimple if $A = A^2$ for every ideal A of S (that is, every ideal is idempotent).

A left (right) ideal A is finitely generated if

$$A = \bigcup_{i=1}^{n} x_i \cup Sx_i \ (A = \bigcup_{i=1}^{n} x_i \cup x_iS).$$

An ideal A is finitely generated if

$$A = \bigcup_{i=1}^{n} x_i \cup x_iS \cup Sx_i \cup Sx_iS$$

where $x_i, \ldots, x_n \in S$ are a finite number of generators of A. In particular, if

$$A = x \cup Sx, A = x \cup xS \text{ and } A = x \cup xS \cup Sx \cup SxS,$$

then A is called a principal left ideal, principal right ideal and principal ideal. For any $x \in S$, the principal left ideal, principal right ideal and principal ideal generated by x is denoted by S^1x, xS^1 and S^1xS^1.

Let S be a semigroup. Then S is said to be left (right) cancellative if

$$ax = ay \Longrightarrow x = y \ \ (xa = ya \Longrightarrow x = y)$$

for all $a, x, y \in S$. A semigroup S is called cancellative if S is both left and right cancellative.

Let S be a semigroup. An ideal P of S is a prime ideal whenever A and B are any two ideals with $AB \subseteq P$ and $A \nsubseteq P$, then $B \subseteq P$. Suppose A is an ideal contained in P. Then P is a minimal prime divisor of A whenever $A \subseteq Q \subseteq P$ for some prime ideal Q, then $Q = P$. Radical of an ideal A, denoted by $\sqrt{A}$, is the intersection of all its minimal prime divisors. An ideal Q is a primary ideal whenever A and B are ideals with $AB \subseteq Q$ and $A \nsubseteq Q$, then $B \subseteq \sqrt{Q}$. Further, Q is P-primary for any prime ideal P, if Q is primary with $\sqrt{Q} = P$. If every ideal of S is finitely generated and principal, then S is called a Noetherian semigroup and principal ideal semigroup, respectively. A semigroup S is called regular if for every $a \in S$, $a = asa$ for some $s \in S$. The semigroup S is said to be (left, right, intra-) regular if for each $a \in S$, there exist $s, t \in S$ such that ($a = sa^2$, $a = a^2s$, $a = sa^2t$). A semigroup S is called Q^*-simple if it contains no proper prime ideals. Clearly, Q^*-simple semigroups are archimedian semigroups as defined in (Petrich, 1973, p. 49).

Let $P_0 < P_1 < \ldots < P$ of $r + 1$ proper prime ideals of S be a chain. The length of such a chain is the integer r. The dimension of S, denoted by dim S, is the supremum of the lengths of all chains of distinct proper prime ideals of S. If S contains no proper prime ideals, then dim S is undefined. It is noted in (Mannepalli, 1975) that dim $S = \infty$ or n where n is a non-negative integer. If P is a minimal prime divisor of an ideal A of S, then the intersection of all P-primary ideals containing A is called the isolated P-primary component of A. The intersection of all the isolated P-primary components of A, as P runs through the minimal prime divisors of A, is called Ker A.

2.2 Γ-Semigroups

Let $S = \{a, b, c, \dots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \dots\}$ be two non-empty sets. Then S is called a Γ-semigroup if there exists a mapping $S \times \Gamma \times S \longrightarrow S \mid (a, \alpha, b) \longrightarrow a\alpha b \in S$ satisfying $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$. Let S be an arbitrary semigroup and Γ any non-empty set. Define $S \times \Gamma \times S \longrightarrow S$ by $a\alpha b = ab$ for all $a, b \in S$ and $\alpha \in \Gamma$. Then S is a Γ-semigroup. Thus, a semigroup can be considered to be a Γ-semigroup. However, any Γ-semigroup need not be a semigroup. For instance, let $S = \{-i, i, 0\}$ and $\Gamma = S$. Then S is a Γ-semigroup with respect to multiplication of complex numbers, whereas S does not reduce to a semigroup with respect to multiplication of complex numbers.

A non-empty subset A of Γ-semigroup S is said to be a Γ-subsemigroup of S if $A\Gamma A \subseteq A$. If A and B are two non-empty subsets of a Γ-semigroup S, then $A\Gamma B$ is defined as $A\Gamma B = \{a\gamma b \mid a \in A,\ b \in B \text{ and } \gamma \in \Gamma\}$. For simplicity we write $a\Gamma B$, $A\Gamma b$ and $A\gamma B$ in place of $\{a\}\Gamma B$, $A\Gamma\{b\}$ and $A\{\gamma\}B$ respectively.

Let S be a Γ-semigroup. A non-empty subset A of S is called a left (right) Γ-ideal (or simply a left (right) ideal) of S if $S\Gamma A \subseteq A$ $(A\Gamma S \subseteq A)$. Further, a non-empty subset A of a Γ-semigroup S is called a Γ-ideal if A is both a left and a right Γ-ideal of S. A left (right) Γ-ideal A of a Γ-semigroup S is called idempotent if $A\Gamma A = A$. A left (right) Γ-ideal A of S is called a proper left (right) Γ-ideal of S if $A \neq S$. A proper left (right) Γ-ideal A of S is called a maximal left (right) Γ-ideal of S if A is not contained in any proper left (right) Γ-ideal of S. A Γ-semigroup S is called left (right) simple if it contains no proper left (right) Γ-ideal, that is, for every $a \in S$, $S\Gamma a = S$ $(a\Gamma S = S)$. A Γ-semigroup S is said to be semisimple if $\langle a\rangle\Gamma\langle a\rangle = \langle a\rangle$ for each element a of S.

A Γ-semigroup S is called regular if for each $a \in S$, there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$. Now, S is termed (left, right, intra-) regular if for each $a \in S$, there exist $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that

$$(a = x\alpha a\beta a,\ a = a\alpha a\beta x,\ a = x\alpha a\beta a\gamma y).$$

Let S be a Γ-semigroup. An element a of a Γ-semigroup S is called an α-idempotent if $a\alpha a = a$ for some $\alpha \in \Gamma$. The set of all α-idempotents is denoted by E_α and we denote $\bigcup_{\alpha\in\Gamma} E_\alpha$ by $E(S)$. The elements of $E(S)$ are called idempotent elements of S. Every element of $E(S)$ is called an idempotent element of S. In a regular Γ-semigroup S, we have that $E(S)$ is a non-empty set.

A Γ-semigroup S is said to be left (right) cancellative if

$$a\alpha x = a\alpha y \Longrightarrow x = y \quad (x\beta b = y\beta b \Longrightarrow x = y)$$

for all $a, b, x, y \in S$, $\alpha, \beta \in \Gamma$. A Γ-semigroup S is called cancellative if S is both left and right Γ-cancellative.

It is well known that for each element a of a Γ-semigroup S, the left Γ-ideal $a \cup S\Gamma a$ containing a is the smallest left Γ-ideal of S containing a. If A is any other left Γ-ideal containing a, then $a \cup S\Gamma a \subseteq A$. This is denoted by $\langle a\rangle_l$ and called the principal left Γ-ideal generated by the element a. Similarly, for each $a \in S$, the smallest right Γ-ideal containing a is $a \cup a\Gamma S$ which is denoted by $\langle a\rangle_r$ and called the principal right Γ-ideal generated by the element a. The principal Γ-ideal of S generated by the element a is denoted by $\langle a\rangle$ and $\langle a\rangle = a \cup S\Gamma a \cup a\Gamma S \cup S\Gamma a\Gamma S$.

3. COMMUTATIVE MULTIPLICATION SEMIGROUPS

A commutative semigroup S earns the title of a multiplication semigroup when it adheres to the rule that whenever two ideals A and B of S are such that A is a subset of B, there exists another ideal C such that A equals the product of B and C. That is, $A = BC$. The commutative semigroup under consideration is not a group. Below, we provide some illustrative examples to aid readers in grasping this concept.

(i) The set of integers modulo n with respect to multiplication is a multiplication semigroup.

(ii) Let $S = \{1, x, x^2, \dots, e, f, ef\}$ with $ex = x = fx = efx,\ e = e^2,\ f = f^2$. It can be easily verified that S is a multiplication semigroup.

The following result by (Mannepalli, 1975a) presents the structure of multiplication semigroups containing identity element in the commutative case.

Proposition 3.1. In a multiplication semigroup S, the following assertions are true.

(i) $S = S^2$, $A = AS$ for any ideal A, $aS^1 = aS$ for every $a \in S$.

(ii) If S contains a cancellable element, then S contains identity.

(iii) If P is a prime ideal which is properly contained in an ideal A, then $P = PA$; moreover, $P = A^w$ or $P = PA^w$.

(iv) If P is a prime ideal with $P^n \neq P^{n+1}$ for any positive integer n, then P^w is a prime ideal.

(v) Every ideal with prime radical is primary ideal; in particular, if P is a prime ideal, then P^n is a P-primary ideal for any positive integer n.

(vi) Every primary ideal is some power of its radical.

(vii) If P is proper prime ideal and A is any ideal such that $A \subseteq P^n$, $A \subseteq P^{n+1}$ for some positive integer n, then $P^n = A : yS$ for some $y \in S \backslash P$.

(viii) If A is an idempotent ideal of S, then A is a multiplication semigroup; in particular, for any $x \in A$ we have $x = xy$ for some $y \in A$.

(ix) Every homomorphic image o f S is a multiplication semigroup.

(x) If S is noetherian, then $A = \mathrm{Ker}\ A$ for every ideal A of S.

Theorem 3.1. If S is a regular semigroup or a principal ideal semigroup, then S is a multiplication semigroup.

It was noted in (Mannepalli, 1975a), that the converse to Theorem 3.1 need not be true by the above example (ii). Indeed, S is a multiplication semigroup. However, it is neither a regular semigroup nor a principal ideal semigroup.

(Mannepalli, 1975a) observed that regular semigroups and principal ideal semigroups containing identity elements form subclasses of multiplication semigroups. However, the aforementioned subclasses do not exhaust all multiplication semigroups containing identity. This observation motivated the author to establish the conditions under which multiplication semigroups are regular or principal ideal semigroups. Meanwhile, the conditions for regularity in the case of arbitrary multiplication semigroups could not be deduced. Nonetheless, the conditions for multiplication semigroups in which every ideal is a product of prime ideals to be regular were provided. This class of semigroups includes all noetherian semigroups and some non-noetherian semigroups.

This observation leads to the following theorem and corollary.

Theorem 3.2. Let S be a multiplication semigroup in which every ideal is a product of primary ideals. Then S is regular if and only if every prime ideal is idempotent.

Corollary 3.1. A noetherian multiplication semigroup S is a regular semigroup if and only if every prime ideal is idempotent.

Theorem 3.3. Let S be a multiplication semigroup satisfying any one of the following conditions.

(i) S contains no idempotent prime ideals.

(ii) Every idempotent prime ideal of S contains at most a finite number of prime ideals properly.

(iii) No idempotent prime ideal is equal to the union of all the prime ideals properly contained in it.

(iv) Every idempotent prime ideal of S is finitely generated. Then S is a principal ideal semigroup if and only if S contains identity and the set of prime ideals of S is linearly ordered.

Theorem 3.4. For a finite-dimensional multiplication semigroup S, the following are equivalent.

(i) Every idempotent prime ideal is principal.

(ii) S contains identity and the set of prime ideals of S is linearly ordered.

(iii) S is a principal ideal semigroup with identity.

Theorem 3.5. If S is a finite-dimensional semigroup, then S is multiplication semigroup if and only if S is any one of the following types.

(i) S is a Dedekind semigroup containing only cancellable elements.

(ii) S is a principal ideal semigroup.

(iii) There exists an idempotent prime ideal P which is not principal, which is a multiplication subsemigroup of S (P may or may not contain identity) such that every ideal of S not contained in P is principal.

(Mannepalli, 1975b) observed that most of the facts mentioned and proved for multiplication semigroups in (Mannepalli, 1975a) are also true for those semigroups not containing identity, and some additional structure theorems were included.

Lemma 3.1. If S is a multiplication semigroup, then $aS^1 = aS$ for every $a \in S$ and $S = S^2$.

It was noted by (Mannapalli, 1975b) that if S is a semigroup not containing identity, then there is a one-to-one correspondence between (principal, prime, primary) ideals of S and proper (principal, prime, primary) ideals of S^1.

Proposition 3.2. S is a multiplication semigroup if and only if S^1 is a multiplication semigroup with $S = S^2$.

In view of the above Proposition, Proposition 3.1 holds for any semigroup S not containing identity.

Now, the following result describes multiplication semigroups by employing some ideal-theoretic properties mentioned in section 2.

Lemma 3.2. Let S be a semigroup in which every ideal with prime radical is primary and every primary ideal is some power of its radical. If A is an ideal of S with a minimal prime divisor P, then the isolated P-primary component of A is some power of P.

Theorem 3.6. Let S be a semigroup in which every ideal has a finite number of distinct minimal prime divisors. Then S is a multiplication semigroup if S satisfies the following properties.

(i) $S = S^2$.

(ii) Every primary ideal of S is some power of its radical.

(iii) $A = \text{Ker } A$ for every ideal A of S.

Theorem 3.7. Let S be a semigroup satisfying any one of the following.

(i) Every ideal of S has a finite number of distinct minimal prime divisors and is a product of its minimal prime divisors.

(ii) $\dim S^1 \leq 1$ and every ideal of S has a finite number of distinct minimal prime divisors.

(iii) Prime ideals of S are maximal and every ideal has a finite number of distinct minimal prime divisors. Then S is a multiplication semigroup if and only if S satisfies (i), (ii), (iii) of Theorem 3.6.

Theorem 3.8 A noetherian semigroup S is a multiplication semigroup if and only if S satisfies (i), (ii), (iii) of Theorem 3.6.

4. NON-COMMUTATIVE MULTIPLICATION SEMIGROUPS

The preceding definition by (Mannepalli, 1975a) depicts the commutative case. Moreover, (Mannepalli & Satyanarayana, 1976) defined multiplication semigroup in the non-commutative case as follows:

A semigroup S is called a right multiplication semigroup if for any pair of right ideals A and B of S with $A \subseteq B$, there exists a right (left) ideal C of S such that $A = CB$. By the dual, left multiplication semigroup is defined. The example below explicates the notion of a right multiplication semigroup.

Let $S = \{a, b, c, d\}$ with $aa = ab = ac = ad = a = ba = bd = ca = cd$, $bb = b = bc$, $cb = c = cc$, $da = db = d = dc = dd$. Let $A = \{a\}$, $B = \{a, d\}$, $C = \{a, c\}$. Then A, B and C are all right ideals of the semigroup S. It is not difficult to see that S is a right multiplication semigroup.

Proposition 4.1. For a right multiplication semigroup, the following are true.

(i) $S = S^2$ and $a \in aS$ for every $a \in S$.

(ii) For any right ideal A, $A \subseteq SA$.

(iii) If M is a maximal right ideal, then $M = M^2$ or $M = SM$.

It was noted in (Mannepalli & Satyanarayana, 1976) that if S is a left cancellative right multiplication semigroup, then Proposition 4.1 (i) implies the existence of idempotents. Therefore, the semigroups of Baer-Levi type, which are left simple and left cancellative, are not right multiplication semigroups.

Proposition 4.2. In a semigroup, if $a \in aSaS$ for every $a \in S$, then S is a right multiplication semigroup.

Corollary 4.1. If S is a right regular semigroup or a regular semigroup, then S is a right multiplication semigroup.

(Mannepalli & Satyanarayana, 1976) also noted that the semigroups of Baer-Levi type which are left regular are not right multiplication semigroups.

Theorem 4.1. Let S be a simple semigroup. Then S is a right multiplication semigroup if and only if $a \in aS$ for every $a \in S$.

Theorem 4.2. A semisimple semigroup S is a right multiplication semigroup if and only if $a \in aSaS$ for every $a \in S$.

The next theorem is a consequence of Theorem 4.2.

Theorem 4.3. If S is a semisimple right multiplication semigroup such that $ab \in Sba$ for every $a, b \in S$, then S is a regular semigroup.

Theorem 4.4. For a left cancellative semigroup S, the following are equivalent.

(i) Set S is a semisimple right multiplicatipn semigroup.

(ii) Set S is a simple semigroup with $a \in aS$ for every $a \in S$.

Theorem 4.5. Let every right ideal be a two-sided ideal in a semigroup S. Then S is right regular if and only if S is a semisimple right multiplication semigroup.

Theorem 4.6. The following are equivalent for a semigroup S.

(i) S is a left simple right multiplication semigroup.

(ii) S is a left simple semigroup containing idempotents.

(iii) S is a left group.

Theorem 4.7. If every left ideal is two-sided in a semigroup S, then S is the union of groups if and only if S is a semisimple right multiplication semigroup.

Theorem 4.8. Let S be a left regular or intra-regular semigroup. Then S is a right multiplication semigroup if and only if S is a union of simple semigroups S_α with $x \in xS_\alpha$ for every $x \in S_\alpha$.

In the following (Mannepalli and Satyanarayana, 1976) gives a description of the structure of left cancellative right multiplication semigroups. This generalises the result of (Dorofeeva *et al*., 1975) in the commutative case.

Proposition 4.3. Let S be a semigroup with a unique maximal right ideal M such that $A \neq AM$ for every proper right ideal A, and suppose that every proper right ideal is included in M. Then S is a right multiplication semigroup if and only if S satisfies the following conditions:

(i) Every proper right ideal is of the form M^r,

(ii) $S = S^2$,

(iii) $M = SM$ and

(iv) $M = MS$.

Furthermore, $M = xS^1$ for some $x \in S$ and every element a of M is of the form $x^r u$ for some natural number r, where $u \notin M$.

Theorem 4.9. Let S be a left cancellative semigroup. Then S is a right multiplication semigroup if and only if S contains idempotents and S is one of the following:

(i) S is a simple semigroup with $a \in aS$ for every $a \in S$.

(ii) S contains a unique maximal right ideal M, which is also the unique maximal two-sided ideal, with $M = MS$ such that every proper right ideal is of the form M^r and thus two-sided.

5. MULTIPLICATION Γ-SEMIGROUPS

Let S be a non-commutative Γ-semigroup. Then S is called a right multiplication Γ-semigroup if for any two right Γ-ideals A, B of S such that $A \subseteq B$, there exists a right Γ-ideal C of S such that $A = C\Gamma B$. Analogously, left multiplication Γ-semigroup can be defined. The example given by (Awolola & Ibrahim, 2023) is constructed to bring the idea into focus.

Let $S = \{1, -1, i, -i\}$ be a semigroup under the operation given by the table below:

$\cdot$	1	-1	i	$-i$
1	1	1	1	1
-1	1	-1	-1	1
i	1	i	i	1
$-i$	$-i$	$-i$	$-i$	$-i$

Let $\Gamma = \{\alpha\}$. Define $a\alpha b = ab$. As $a\alpha(b\alpha c) = (a\alpha b)\alpha c$ for all $a, b, c \in S$, S is a Γ-semigroup. Let $A = \{1\}, B = \{1, -i\}$, and $C = \{1, i\}$. Then A, B and C are all right Γ-ideals of the Γ-semigroup S. It is easy to verify that S is a right multiplication Γ-semigroup.

It was noted by (Awolola and Ibrahim, 2023) that, in a right multiplication Γ-semigroup S, $A = A\Gamma B$ for every right Γ-ideal A of S.

Proposition 5.1. Let S be a right multiplication Γ-semigroup. Then

(i) $S\Gamma S = S$.

(ii) $x \in x\Gamma S$ for each $x \in S$.

Proposition 5.2. Let S be right multiplication Γ-semigroup. Then $A \subseteq S\Gamma A$ for any right Γ-ideal A of S.

Proposition 5.3. Let S be a right multiplication Γ-semigroup. If M is a maximal right Γ-ideal containing every proper right Γ-ideal of S, then $M = M\Gamma M$ or M is a Γ-ideal of S such that $M = S\Gamma M$.

Proposition 5.4. Let S be a right multiplication Γ-semigroup and M be a maximal right Γ-ideal containing every proper right Γ-ideal of S. If M is unique such that $M \neq M\Gamma M$, then $M = x\Gamma S$ for some $x \in M \setminus M\Gamma M$.

Proposition 5.5. Let S be a right multiplication Γ-semigroup. If S contains a left cancellative element, then S contains a β-idempotent which is a left identity.

Proposition 5.6. Let S be a right multiplication Γ-semigroup. If S is left Γ- and right Γ-cancellative, then S contains an identity and every right Γ-ideal is a Γ-ideal.

The following result shows some classes of Γ-semigroups satisfying right Γ-multiplication semigroups.

Proposition 5.7. Let S be a Γ-semigroup. If $a \in (a\Gamma S)\Gamma(a\Gamma S)$ for every $a \in S$, then S is a right multiplication Γ-semigroup.

Proposition 5.8. Let S be a Γ-semigroup. If S is regular (right regular), then S is a right multiplication Γ-semigroup.

Proposition 5.9. Let S be a Γ-semigroup. If S is simple with $a \in a\Gamma S$ for every $a \in S$, then S is a right multiplication Γ -semigroup.

Proposition 5.10. Let S be a Γ-semigroup. If S is a left simple right multiplication Γ-semigroup, then S is a left simple Γ-semigroup containing idempotents.

Proposition 5.11. Let S be a right multiplication Γ-semigroup. If S is intra-regular, then S is semisimple.

Proposition 5.12. Suppose that S is a Γ-semigroup such that every right Γ-ideal is a Γ-ideal. Then S is a semisimple right multiplication Γ-semigroup if and only if S is right regular.

Proposition 5.13. Let S be a left cancellative Γ-semigroup. If S is a semisimple right multiplication Γ-semigroup, then S is a simple Γ-semigroup such that $a \in a\Gamma S$ for each $a \in S$.

6. RESEARCH PROBLEMS

In this section, we present some research problems prompted by results reported in the previous section.

(i) Find necessary and sufficient conditions for a semisimple right multiplication Γ-semigroup to be regular.

(ii) Characterise right multiplication Γ-semigroups in terms of some ideal-theoretic property other than the defining property.

(iii) Construct larger right multiplication Γ-semigroups by considering the direct products or extensions.

The natural questions in this direction are:

(a) When can "a direct product of two right multiplication Γ-semigroups" be a right multiplication Γ-semigroup?

(b) When is "an extension of a right multiplication Γ-semigroup by a right multiplication Γ-semigroup" again a right multiplication Γ-semigroup?

(iv) Develop the representation theory for right multiplication Γ-semigroups.

(v) When do right multiplication Γ-semigroups admit a Γ-ring structure?

It is essential to note that the existing results, together with the open problems presented in sections 5 and 6, for right multiplication Γ-semigroups, are practicable for left multiplication Γ-semigroups.

7. CONCLUSION AND SUGGESTION

An overview of the development of multiplication Γ-semigroups with some research problems outlined is presented. It is found that Γ-multiplication property is not only enjoyed by regular Γ-semigroups but also by other classes of Γ-semigroups.

The suggested investigations are feasible for extension in a fuzzy framework for future research work.

REFERENCES

Awolola, J. A. and Ibrahim, M. A., *Multiplication Γ-semigroups*, Jordan J. Math. Stat., **16**(3) (2023), 411-420.

Ibrahim, M. A. and Awolola, J. A., *On Concept of Γ-semigroups*, Algebras, Groups and Geometries, **39**(2) (2023), 161-173.

Awolola, J. A. and Ibrahim, M. A., *Connections of a Green's Relations of A Γ-semigroups with Operator Semigroups*, Annals Math. Comp. Sci., **22**(3) (2024), 411-420.

Clifford, A. H. and Preston, G. B., *The Algebraic Theory of Semigroups*, Vol. I, Math. Surveys No. 7, Amer. Math. Soc., Providence, 1961.

Clifford, A. H. and Preston, G. B., *The Algebraic Theory of Semigroups*, Vol. II, Math. Surveys No. 7, Amer. Math. Soc., Providence, 1967.

Dorofeeva, M. P., Mannepalli, V. L. and Satyanarayana, M., *Prüfer and Dedekind Monoids*, Semigroup Forum, **9** (1975), 294-309.

Dutta, T. K. and Chattopadhyay, S., *On Uniformly Strongly Prime Γ-semigroup*, Analele Stiintifice Ale Universitatii" AL. I. CUZA" IASI, Tomul LII (2006), f.2, 325–335.

Gilmer, R. W. and Mott, J. L., *Multiplication Rings as Rings in which Ideals with Prime Radicals Are Primary*, Trans. Amer. Math. Soc., 114 (1965), 40-52.

Griffin, M., *Multiplication Rings via their Total Quotient Rings*, Can. J. Math., **26** (1974), 430-449.

Krull, W., *Idealtheorie, Ergebn*. Math. IV, New York, 1948.

Larsen, M. and McCarthy, P., *Multiplicative Theory of Ideals*, Academic Press, New York, 1971.

Mannepalli, V. L., *Multiplication Semigroup*, Semigroup Forum, **11** (1975), 310-327.

Mannepalli, V. L., *Multiplication Semigroup II*, Semigroup Forum, **12** (1975), 145-164.

Mannepalli, V. L. and Satyanarayana, M., *Multiplication Semigroup III*, Semigroup Forum, **12** (1976), I65-175.

Mori, S., *Uber Allgemeine Multiplikationsringe. I*, J. Sci. Hiroshima Univ. Ser. A-I Math., **4** (1934), 1-26.

Mott, J. L., *Equivalent Conditions for A Ring to be Multiplication Ring*, Can. J. Math., **16** (1964), 429-434.

Mott, J. L., *Multiplication Rings Containing Only Finitely Many Minimal Prime Ideals*, J. Sci. Hiroshima Univ. Ser., **16** (1969), 73-83.

Petrich, M., *Introduction to Semigroups*, Merrill Publ. Co., Columbus, Ohio, 1973.

Saha, N. K., *On* Γ-*semigroups II*, Bull. Cal. Math. Soc., **79** (1987), 331-335.

Sen, M. K. *On* Γ-*semigroups*, Proc. Int. Conf. Algebra Appl., Decker Publisher, New York, 1981, 301–308.

Sen, M. K. and Saha, N. K., *On* Γ-*semigroups I*, Bull. Cal. Math. Soc., **78** (1986), 180-186.

Sen, M. K. and Saha, N. K., *Orthodox* Γ-*semigroups*, Int. J. Mat. Math. Soc., **13**(3) (1990), 527-534.

Sen, M. K. and Chattopadhyay, S., *Semidirect Product of A Monoid and A* Γ-*semigroups*, East-West J. Math., 6(2) (2004), 131-138.

Sen, M. K. and Chattopadhyay, S., Γ-*semigroups: A Survey*, *Algebra and its Applications*, **174** (2016), 225-239.

Seth, A., Γ-*group Congruences on Regular* Γ-*semigroups*, Int. J. Mat. Math. Sci., **15**(1) (1992), 103-106.