THE WEAK SUBSTITUTION IN COMPLEX NUMBERS OVER THE SYMMETRIZED MAX-PLUS ALGEBRA

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Abstract. The complex numbers over the symmetrized max-plus algebra is an extension of numbers in the symmetrized max-plus algebra. This complex number is formed by defining an imaginary form in the symmetrized max-plus algebra. The substitution process in complex numbers over the symmetrized max-plus algebra is a very important concept and cannot be done as in complex numbers over the conventional algebra. Thus, several algebraic properties are needed to ensure that the substitution process can be carried out. In this paper, we show the substitution rules in complex numbers over the symmetrized max-plus algebra. These rules is formulated by extending the weak substitution property in the symmetrized max-plus algebra. The extension was carried out in the symmetrized max-plus algebra part and imaginary part of complex numbers. The results obtained are the weak substitution in complex numbers over the symmetrized max-plus algebra. Substitution process can be carried out in complex numbers if both of the symmetrized max-plus algebra part and imaginary part of complex numbers are signed elements. Then, according the weak substitution property, it can be derived the weak transitive property in complex numbers over the symmetrized max-plus algebra. Furthermore, it can be also derived reduction of balance property in complex numbers over the symmetrized max-plus algebra, which reduce balance relation into equal relation sense.

Keywords: balance, complex numbers, substitution, symmetrized max-plus algebra

A. Introduction

The max-plus algebra is defined as the set $\mathbb{R} \cup \{-\infty\}$ with operation of maximum (denoted as "max") for addition and usual addition (denoted as "plus") for multiplication, where $\mathbb{R}$ represents the set of all real numbers. This algebraic structure is denoted by $\mathbb{R}_{\max}$. It differs from conventional algebra in that not every element in the max-plus algebra has an additive inverse, except for the zero element [1,2,4,8,9].

The symmetrization process can be employed to solve the absence of additive inverses. This process utilizes a balance relation (denoted by ∇) to determine additive inverse-like of the elements in the max-plus algebra. The outcome of this symmetrization process is referred to as the symmetrized max-plus algebra, denoted by $\mathbb{S}$. Consequently, $\mathbb{R}_{\max}$ can be seen as the positive or zero parts of $\mathbb{S}$ [5,6,7].

In the discussion of conventional algebra, the substitution property has many important roles in algebra operations. If $x = a$ and $cx = b$ then $ca = b$ always applies. The substitution property of conventional algebra in equation problems does not fully apply to balances problems for discussing in the symmetrized max-plus algebra. For example, given $5^{\bullet}\nabla 2$ and

$3 \otimes 5^{\bullet}\nabla 8$,, but $3 \otimes 2$ is not balanced with 8. The substitution property in the symmetrized max-plus algebra was introduced in [4], and is hereafter called the weak substitution property.

The complex numbers over the symmetrized max-plus algebra is an extension of numbers in the symmetrized max-plus algebra. This complex number is formed by defining an imaginary number form in the symmetrized max-plus algebra. Thus, in general, the symmetrized max-plus algebraic numbers can be viewed as special occurrences of complex numbers over the symmetrized max-plus algebra.

In this article, we present a discussion of the extension of the substitution property to complex numbers over the symmetrized max-plus algebra, by adopting the weak substitution property in [4]. This substitution rule needs to be discussed because it guarantees the validity of operations involving substitution on complex numbers over the symmetrized max-plus algebra which cannot be directly adopted from conventional algebraic discussions.

B. Methods

This research is a literature research that examines and develops research that has been done previously, i.e substitution process in complex numbers over the symmetrized max-plus algebra. The step procedures taken in this research are presented in Figure 1.

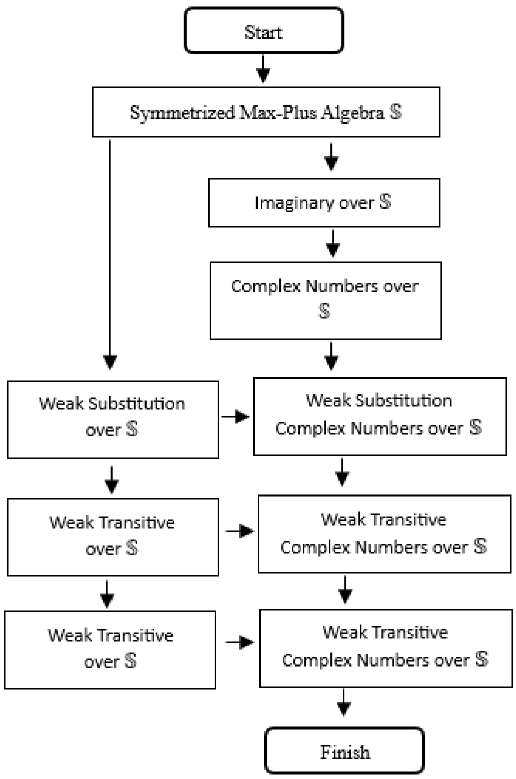

Figure 1. Step Procedure of Research

C. Results And Discussion

This section is the main part of this research which contains construction of substitution rules in symmetrized max-plus algebra and complex numbers over symmetrized max-plus algebra.

1. Substitution in the Symmetrized Max-Plus Algebra

When discussing the symmetrized max-plus algebra, the substitution property cannot be used as in linear algebra. In the discussion of linear algebra, if $x = a$ and $cx = b$ then by substituting $x = a$ in the equation $cx = b$ we obtain $ca = b$. This substitution property does not always apply to balance elements in the symmetrized max-plus algebra. If $x\nabla a$ and $c \otimes x\nabla b$ then by substituting $x\nabla a$ in the equation $c \otimes x\nabla b$ we do not always get $c \otimes a\nabla b$. For example, given $5^{\bullet}\nabla 2$ and $3 \otimes 5^{\bullet}\nabla 8$, but $3 \otimes 2 \not\nabla 8$.

The following theorem explains the substitution rule in the symmetrized max-plus algebra.

Theorem 1. *If $x\nabla a$ and $c \otimes x\nabla b$ where $x \in \mathbb{S}^{\vee}$, then $c \otimes a\nabla b$.*

Proof. Since $x \in \mathbb{S}^{\vee}$, we have either $x \in \mathbb{S}^{\oplus}$ or $x \in \mathbb{S}^{\ominus}$.

Case 1: For $x \in \mathbb{S}^{\oplus}$, let $x = \overline{(x', -\infty)}$, $a = \overline{(a_1, a_2)}$, $b = \overline{(b_1, b_2)}$ and $c = \overline{(c_1, c_2)}$. According to $x\nabla a$ and $c \otimes x\nabla b$, we have $x' \oplus a_2 = a_1$ and $c_1 \otimes x' \oplus b_2 = c_2 \otimes x' \oplus b_1$. Then by adding $c_1 \otimes a_2 \oplus c_2 \otimes a_2$ to the last equality, we have

$$c_1 \otimes x' \oplus b_2 \oplus c_1 \otimes a_2 \oplus c_2 \otimes a_2$$
$$= c_2 \otimes x' \oplus b_1 \oplus c_1 \otimes a_2 \oplus c_2 \otimes a_2$$

so

$$c_1 \otimes (x' \oplus a_2) \oplus b_2 \oplus c_2 \otimes a_2$$
$$= c_2 \otimes (x' \oplus a_2) \oplus b_1 \oplus c_1 \otimes a_2.$$

By using $x' \oplus a_2 = a_1$, it yields

$$c_1 \otimes a_1 \oplus c_2 \otimes a_2 \oplus b_2 = c_2 \otimes a_1 \oplus c_1 \otimes a_2 \oplus b_1$$

and consequently

$$\overline{(c_1 \otimes a_1 \oplus c_2 \otimes a_2, c_1 \otimes a_2 \oplus c_2 \otimes a_1)} \otimes \overline{(a_1, a_2)}$$
$$= \overline{(c_1, c_2)} \otimes \overline{(a_1, a_2)} \nabla \overline{(b_1, b_2)}.$$

The last balances is $c \otimes a\nabla b$.

Case 2: For $x \in \mathbb{S}^{\ominus}$, let $x = \overline{(-\infty, x')}$, $a = \overline{(a_1, a_2)}$, $b = \overline{(b_1, b_2)}$ and $c = \overline{(c_1, c_2)}$. According to $x\nabla a$ and $c \otimes x\nabla b$, we have $x' \oplus a_1 = a_2$ and $c_2 \otimes x' \oplus b_2 = c_1 \otimes x' \oplus b_1$. Then by adding $c_1 \otimes a_1 \oplus c_2 \otimes a_1$ to the last equality, we have

$$c_2 \otimes x' \oplus b_2 \oplus c_1 \otimes a_1 \oplus c_2 \otimes a_1$$
$$= c_1 \otimes x' \oplus b_1 \oplus c_1 \otimes a_1 \oplus c_2 \otimes a_1$$

and so

$$c_2 \otimes (x' \oplus a_1) \oplus b_2 \oplus c_1 \otimes a_1$$
$$= c_1 \otimes (x' \oplus a_1) \oplus b_1 \oplus c_2 \otimes a_1.$$

By using $x' \oplus a_1 = a_2$, it yields

$$c_2 \otimes a_2 \oplus c_1 \otimes a_1 \oplus b_2 = c_1 \otimes a_2 \oplus c_2 \otimes a_1 \oplus b_1$$

and consequently

$$\overline{(c_1 \otimes a_1 \oplus c_2 \otimes a_2, c_1 \otimes a_2 \oplus c_2 \otimes a_1)} \otimes \overline{(a_1, a_2)}$$
$$= \overline{(c_1, c_2)} \otimes \overline{(a_1, a_2)} \nabla \overline{(b_1, b_2)}.$$

The last balances is $c \otimes a\nabla b$. ■

For example, let $2\nabla 4^{\bullet}$ and $3 \otimes 2 \nabla 5$ where $2 \in \mathbb{S}^{\vee}$, then $3 \otimes 4^{\bullet} \nabla 5$. The substitution rule in the symmetrized max-plus algebra is called “weak” substitution.

The following theorem explains the transitive rule in the symmetrized max-plus algebra.

Theorem 2. If $a\nabla x$ and $x\nabla b$ where $x \in \mathbb{S}^{\vee}$, then $a\nabla b$.

Proof. Since $x \in \mathbb{S}^{\vee}$, we have either $x \in \mathbb{S}^{\oplus}$ or $x \in \mathbb{S}^{\ominus}$.

Case 1 : For $x \in \mathbb{S}^{\oplus}$, let $x = \overline{(x', -\infty)}$, $a = \overline{(a_1, a_2)}$ and $b = \overline{(b_1, b_2)}$. Since $a\nabla x$ and $x\nabla b$, we have

$a_1 = a_2 \oplus x'$ and $x' \oplus b_2 = b_1$.

Then by adding a_2 to the last equality, we have

$$a_2 \oplus x' \oplus b_2 = a_2 \oplus b_1.$$

By using $a_1 = a_2 \oplus x'$, it yields

$$a_1 \oplus b_2 = a_2 \oplus b_1.$$

The last equality is $a\nabla b$.

Case 2: For $x \in \mathbb{S}^{\ominus}$, let $x = \overline{(-\infty, x')}$, $a = \overline{(a_1, a_2)}$ and $b = \overline{(b_1, b_2)}$. Since $a \nabla x$ and $x \nabla b$, we have

$a_2 = a_1 \oplus x'$ and $x' \oplus b_1 = b_2$.

Then by adding a_1 to the last equality, we have

$$a_1 \oplus x' \oplus b_1 = a_1 \oplus b_2.$$

By using $a_2 = a_1 \oplus x'$, it yields

$$a_2 \oplus b_1 = a_1 \oplus b_2.$$

The last equality is $a \nabla b$. ■

For example, given $2 \nabla 4^{\bullet}$ and $2^{\bullet} \nabla 2$ where $2 \in \mathbb{S}^{\vee}$ then $3 \otimes 4^{\bullet} \nabla 5$. The following theorem explains the reduction of balance into equation rule in the symmetrized max-plus algebra.

Theorem 3.If $x \nabla y$ where $x, y \in \mathbb{S}^{\vee}$, then $x = y$.

Proof. Since $x, y \in \mathbb{S}^{\vee}$, we have either $x, y \in \mathbb{S}^{\oplus}$ or $x, y \in \mathbb{S}^{\ominus}$.

Case 1 : For $x \in \mathbb{S}^{\oplus}$, let

$x = \overline{(x', -\infty)}$ and $y = \overline{(y', -\infty)}$.

Since $x \nabla y$, then $x' = y'$. Therefore, $x = y$.

Case 2 : For $x \in \mathbb{S}^{\ominus}$, let

$x = \overline{(-\infty, x')}$ and $y = \overline{(-\infty, y')}$.

Since $x \nabla y$, then $x' = y'$. Therefore, $x = y$. ■

2. Substitution Rules in Complex Numbers over the Symmetrized Max-Plus Algebra

Let $\mathbb{S}$ be the symmetrized max-plus algebra with the zero element $\mathcal{E} = -\infty$ and the unity element $e = 0$. It is defined k such that $k \otimes k = k^2 = \ominus\, 0$, and k is called an imaginary number in the symmetrized max-plus algebra senese. Then, it is defined a number

$$a \oplus b \otimes k$$

where $a, b \in \mathbb{S}$ and $k^2 = \ominus\, 0$. Furthermore, this number is called the complex numbers over the symmetrized max-plus algebra.

Let $\mathbb{T}$ be the set of all complex numbers over the symmetrized max-plus algebra i.e

$$\mathbb{T} = \{a \oplus b \otimes k \mid a, b \in \mathbb{S} \text{ and } k^2 = \ominus\, 0\}.$$

If $a \oplus b \otimes k \in \mathbb{T}$, then a and b are called the symmetrized max-plus algebra part and the imaginary part of the complex number over the symmetrized max-plus algebra, respectively. The zero complex number over the symmetrized max-plus algebra, i.e $a \oplus b \otimes k$ where a and b are the zero element of $\mathbb{S}$, respectively. Therefore, the zero complex number over the symmetrized max-plus algebra is $(-\infty) \oplus (-\infty) \otimes k$, and simply written as $-\infty$. For every $a \in \mathbb{S}$, it can always be expressed in the form $a \oplus (-\infty) \otimes k$, and simply written as a. Thus, $\mathbb{S}$ can be viewed as a subset of $\mathbb{T}$. In other words, the symmetrized max-plus algebra can be viewed as special part of the set of all complex numbers over the symmetrized max-plus algebra. Furthermore, we have the relation

$$\mathbb{R}_{\max} \subset \mathbb{S} \subset \mathbb{T}.$$

We will define some algebraic operations on $\mathbb{T}$ in order to investigate of the algebraic properties of $\mathbb{T}$.

Definition 4. *Let* $a \oplus b \otimes k, c \oplus d \otimes k \in \mathbb{T}$. *The similarity of two elements in* $\mathbb{T}$ *is defined as*

$$a \oplus b \otimes k = c \oplus d \otimes k$$

if $a = c$ and $b = d$.

Definition 5. *Let* $a \oplus b \otimes k \in \mathbb{T}$. *The minus of elements* $a \oplus b \otimes k$ *is defined as*

$$\ominus\,(a \oplus b \otimes k) = (\ominus\, a) \oplus (\ominus\, b) \otimes k.$$

Furthermore, $(\ominus\, a) \oplus (\ominus\, b) \otimes k$ is simply written as $\ominus\, a \ominus b \otimes k$.

Definition 6. *Let* $a \oplus b \otimes k, c \oplus d \otimes k \in \mathbb{T}$. *The balanced of two elements in* $\mathbb{T}$ *is defined as* $a \oplus b \otimes k \,\nabla\, c \oplus d \otimes k$ if and only if $a \nabla c$ and $b \nabla d$.

Noted that $2 \oplus 3 \otimes k \nabla 3^{\bullet} \oplus 3 \otimes k$ since $2 \nabla 3^{\bullet}$ dan $3 \nabla 3$, but $2 \oplus 3 \otimes k \neq 3^{\bullet} \oplus 3 \otimes k$, because of $2 \neq 3^{\bullet}$. Then, $2 \oplus 3 \otimes k = 2 \oplus 3 \otimes k$, since $2 = 2$ and $3 = 3$, but it also satisfies

$$2 \oplus 3 \otimes k \nabla 2 \oplus 3 \otimes k.$$

The "equal" relation satisfies "balance" relation, but the reverse does not apply. Furthermore, an equality is a special case of balance.

Definition 7. *Let $a \oplus b \otimes k, c \oplus d \otimes k \in \mathbb{T}$. The addition of two elements in $\mathbb{T}$ is defined as*

$$(a \oplus b \otimes k) \oplus (c \oplus d \otimes k)$$
$$= (a \oplus c) \oplus (b \oplus d) \otimes k.$$

Definition 8. *Let $a \oplus b \otimes k, c \oplus d \otimes k \in \mathbb{T}$. The multiplication of two elements in $\mathbb{T}$ is defined as*

$$(a \oplus b \otimes k) \otimes (c \oplus d \otimes k)$$
$$= (a \otimes c \ominus b \otimes d) \oplus (a \otimes d \oplus b \otimes c) \otimes k$$

Let $2 \oplus 3 \otimes k, 3^{\bullet} \oplus (-1) \otimes k \in \mathbb{T}$, then we have $(2 \oplus 3 \otimes k) \oplus (3^{\bullet} \oplus (-1) \otimes k) = 3^{\bullet} \oplus 3 \otimes k$, and $(2 \oplus 3 \otimes k) \otimes (3^{\bullet} \oplus (-1) \otimes k) = 5^{\bullet} \oplus 6^{\bullet} \otimes k$.

We construct the weak substitution in complex numbers over the symmetrized max-plus algebra by extending weak substitution in the symmetrized max-plus algebra. The following theorem discuss the weak substitution in complex numbers over the symmetrized max-plus algebra.

Theorem 9. If $(x \oplus y \otimes k) \nabla (a \oplus b \otimes k)$ and

$$(c \oplus d \otimes k) \otimes (x \oplus y \otimes k) \nabla (p \oplus q \otimes k)$$

where $x, y \in \mathbb{S}^{\vee}$, then

$(c \oplus d \otimes k) \otimes (a \oplus b \otimes k) \nabla (p \oplus q \otimes k)$.

Proof. Since $(x \oplus y \otimes k) \nabla (a \oplus b \otimes k)$, we have $x \nabla a$ and $y \nabla b$. Let

$$(c \oplus d \otimes k) \otimes (x \oplus y \otimes k) \nabla (p \oplus q \otimes k)$$

and so

$$(c \otimes x \ominus d \otimes y) \oplus (c \otimes y \oplus d \otimes x) \otimes k$$
$$\nabla (p \oplus q \otimes k).$$

We have $c \otimes x \ominus d \otimes y \nabla p$ and $c \otimes y \oplus d \otimes x \nabla q$.

Since $x \nabla a$, $y \nabla b$, and $c \otimes x \ominus d \otimes y \nabla p$, $x, y \in \mathbb{S}^{\vee}$, we have $c \otimes a \ominus d \otimes b \nabla p$. Because of $x \nabla a$, $y \nabla b$ and $c \otimes y \oplus d \otimes x \nabla q$, where $x, y \in \mathbb{S}^{\vee}$, then

$c \otimes b \ominus d \otimes a \nabla q$.

Consequently,

$$(c \otimes a \ominus d \otimes b) \oplus (c \otimes b \oplus d \otimes a) \otimes k$$
$$\nabla (p \oplus q \otimes k),$$

and so

$$(c \oplus d \otimes k) \otimes (a \oplus b \otimes k) \nabla (p \oplus q \otimes k). \blacksquare$$

Let $3 \oplus 2 \otimes k$, $4^{\bullet} \oplus 2 \otimes k$, $-5 \oplus 2^{\bullet} \otimes k$ and $4 \oplus 3^{\bullet} \otimes k$ is comple numbers over the symeetrized max-plus algebra, respectively. Note that

$(3 \oplus 2 \otimes k) \nabla (4^{\bullet} \oplus 2 \otimes k)$ and

$(-5 \oplus 2^{\bullet} \otimes k) \otimes (3 \oplus 2 \otimes k)$

$= (\ominus 4^{\bullet} \oplus 5^{\bullet} \otimes k) \nabla (4 \oplus 3^{\bullet} \otimes k)$, then

$$(-5 \oplus 2^{\bullet} \otimes k) \otimes (4^{\bullet} \oplus 2 \otimes k)$$
$$= (\ominus 4^{\bullet} \oplus 6^{\bullet} \otimes k) \nabla (4 \oplus 3^{\bullet} \otimes k).$$

The following theorem discuss the transitive rules in complex numbers over the symmetrized max-plus algebra.

Theorem 10. If $(a \oplus b \otimes k) \nabla (c \oplus d \otimes k)$ and

$$(c \oplus d \otimes k) \nabla (x \oplus y \otimes k)$$

where $c, d \in \mathbb{S}^{\vee}$, then $a \nabla b$.

Proof. Since $(a \oplus b \otimes k) \nabla (c \oplus d \otimes k)$ and
$(c \oplus d \otimes k) \nabla (x \oplus y \otimes k)$,
we have $a \nabla c$ and $b \nabla d$, also $c \nabla x$ and $d \nabla y$. Because of $a \nabla c$ and $c \nabla x$ where $c \in \mathbb{S}^{\vee}$, we have $a \nabla x$. Reciprocally, since $b \nabla d$ and $d \nabla y$ where $d \in \mathbb{S}^{\vee}$, we have $b \nabla y$. Consequently, $(a \oplus b \otimes k) \nabla (x \oplus y \otimes k)$■

Let $(2 \oplus 4^{\bullet} \otimes k), (2 \oplus 3 \otimes k), (5^{\bullet} \oplus 3^{\bullet} \otimes k)$ is complex numbers over the symmetrized max-plus algebra, respectively. Note that

$$(2 \oplus 4^{\bullet} \otimes k) \nabla (2 \oplus 3 \otimes k)$$

and
$(2 \oplus 3 \otimes k) \nabla (5^{\bullet} \oplus 3^{\bullet} \otimes k)$,
where $2,3 \in \mathbb{S}^{\vee}$. Consequently,
$(2 \oplus 4^{\bullet} \otimes k) \nabla (5^{\bullet} \oplus 3^{\bullet} \otimes k)$,
because of $2 \nabla 5^{\bullet}$ and $4^{\bullet} \nabla 3^{\bullet}$.

The following theorem discuss the reduction of balances in complex numbers over the symmetrized max-plus algebra.

Theorem 11. If $(a \oplus b \otimes k) \nabla (c \oplus d \otimes k)$ where $a, b, c, d \in \mathbb{S}^{\vee}$, then
$(a \oplus b \otimes k) \nabla (c \oplus d \otimes k)$.

Proof. Since $(a \oplus b \otimes k) \nabla (c \oplus d \otimes k)$, we have $a \nabla c$ and $b \nabla d$. Because of $a \nabla c$ and $b \nabla d$ where $a, b, c, d \in \mathbb{S}^{\vee}$, then $a = c$ and $b = d$. Consequently,

$$(a \oplus b \otimes k) = (c \oplus d \otimes k)■$$

D. Conclusion

The substitution rules in complex numbers over the symmetrized max-plus algebra cannot be applied as in complex numbers in conventional algebra. This substitution rules can be constructed using the weak substitution in the symmetrized max-plus algebra. Then, it can be derived the transitive and reduction of balance in complex numbers over the symmetrized max-plus algebra. The further research can be done in construction of the substitution rule in matrix over the set of complex numbers over the symmetrized max-plus algebra

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