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SOME PROPERTIES OF SUBSEMIHYPERGROUPS

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ABSTRACT. *In this paper we will present two properties of subsemihypergroups. The first property is a relation between subsemihypergroups and semihypergroup. This property enable us to get the second property, which provides a relation between subsemihypergroups and regular semihypergroups.*

Keywords*: semihypergroup, subsemihypergroup, regular*.

ABSTRAK. Pada makalah ini disajikan dua buah sifat subsemihypergrup. Sifat pertama adalah hubungan antara subsemihypergrup dan semihypergrup. Berdasarkan sifat ini, selanjutnya diperoleh sifat kedua yakni hubungan antara subsemihipergrup dan semihipergrup reguler.

Kata Kunci: semihipergrup, subsemihipergrup, reguler

1. INTRODUCTION

Nowadays, many researchers work on algebraic hyperstructure, which is firstly introduced by Marty [10] in 1934. The algebraic hyperstructure can be viewed as a generalization of the algebraic structure. For example, if the algebraic structure offers semigroups, then the algebraic hyperstructure offers the so called semihypergroups. A semihypergroup is a nonempty set which is equipped with a binary hyperoperation and satisfies the associative property. Some results on semihypergroups have been established by Bonansinga and Corsini [1], Davvaz [3], Davvaz and Poursalavati [4], Fasino and Freni [5], Guţan [6], Hasankhani [7], Javarpour, *et al.* [8], and Leoreanu [9].

A semihypergroup may consist of some regular elements. An element *x* in a semihypergroup $(H, \circ)$ is called a regular element if the element *x* satisfies the property $x \in x \circ H \circ x$. This means that there is an element y in H such that we have $x \in x \circ y \circ x$. Here, $\circ$ denotes a binary operation. If every element in *H* is a regular element, then the semihypergroup $(H, \circ)$ is a regular semihypergroup. Some properties of a regular semihypergroup have been provided by Onipchuk [11]. The properties given in [11] have no relation with subsemyhipergroups. Hence, in this paper we will present a relation between regular semihypergroups and subsemihypergroup. This property need another property which relating semihypergroup and subsemihypergroup. Another property regarding the Cartesian product of two subsemihypergroup is a consequence of the second property

The definition of subsemihypergroup is available in Davvaz [2] page 54.

Definition 1.1 (Subsemihypergroup)

Let $(H,\circ)$ is a semihypergroup and A is a nonempy subset of H. The set A is a subsemihypergroup of H if $A \circ A \subseteq A$.

2. RESULTS AND DISCUSSION

The first result in this paper is a property illustrating a relation between subsemihypergroups and semihypergroups. This result is stated in the following theorem.

Lemma 2.1

If $(A, \circ)$ is a subsemihypergroup of semihypergroup $(H, \circ)$, then $(A, \circ)$ is a semihypergroup.

Proof. Since $(A, \circ)$ is a subsemihypergroup of the semihypergroup $(H, \circ)$, then A is a nonempty subset of H, and every element A is also in H. For every $x, y \in A$, we have $x \circ y \in P^*(H)$, where $P^*(H)$ denotes the collection of nonempty subsets of H. As $A \circ A \subseteq A$, then $x \circ y \in P^*(A)$. This means that $\circ$ is a binary hyperoperation on A. Furthermore, as $A \subseteq H$ and the hyperoperation $\circ$ on H is assosiative, we find that the associative property in H is inherited to A. This prove that $(A, \circ)$ is a semihypergroup. ■

Before we come to a property of regular semihypergroups, we firtsly present some examples of regular semihypergroups.

Example 2.1

Let the set of natural numbers $\mathbb{N}$ is eqquipped with a hyperoperation $\circ$ which is defined by

$$x \circ y = \{t \in \mathbb{N} | \ t \geq \text{maks } \{x, y\}\}$$

for every $x, y \in \mathbb{N}$. Here, the set $(\mathbb{N}, \circ)$ is a regular semihypergroup as it satisfies the following properties.

1. Hyperoperation $\circ$ is a binary operation on $\mathbb{N}$ since for every $x, y \in \mathbb{N}$ we have $x \circ y$ is a subset of $\mathbb{N}$. Furthermore, $x \circ y \in P^*(\mathbb{N})$, where $P^*(\mathbb{N})$ denote the collection of nonempty subset of $\mathbb{N}$.
2. For every $x, y, z \in \mathbb{N}$, we have

$$\begin{aligned} x \circ (y \circ z) &= \{x\} \circ \{t \in \mathbb{N} | \ t \geq \text{maks } \{y, z\}\} \\ &= \bigcup_{v \in \{t \in \mathbb{N} | \ t \geq \text{maks } \{y,z\}\}} (\{x\} \circ v) \\ &= \{t \in \mathbb{N} | \ t \geq \text{maks } \{x, \text{maks } \{y, z\}\}\} \\ &= \{t \in \mathbb{N} | \ t \geq \text{maks } \{x, y, z\}\} \end{aligned}$$

$$\begin{aligned} (x \circ y) \circ z &= \{t \in \mathbb{N} | \ t \geq \text{maks } \{x, y\}\} \circ \{z\} \\ &= \bigcup_{u \in \{t \in \mathbb{N} | \ t \geq \text{maks } \{x,y\}\}} (u \circ \{z\}) \\ &= \{t \in \mathbb{N} | \ t \geq \text{maks } \{\text{maks } \{x, y\}, z\}\} \\ &= \{t \in \mathbb{N} | \ t \geq \text{maks } \{x, y, z\}\}. \end{aligned}$$

   This shows us that the hyperoperation is associative.
3. For any $x \in \mathbb{N}$, we can always find $y \in \mathbb{N}$ with $y \leq x$ such that

$$\begin{aligned} x \circ y \circ x &= (x \circ y) \circ x \\ &= \{t \in \mathbb{N} | \ t \geq \text{maks } \{x, y\}\} \circ \{x\} \end{aligned}$$

$= \bigcup_{u \in \{t \in \mathbb{N} \mid t \geq \text{maks}\{x,y\}\}} u \circ \{x\}$

$= \{t \in \mathbb{N} \mid t \geq \text{maks}\,\{\text{maks}\,\{x, y\}, x\}\}$

$= \{t \in \mathbb{N} \mid t \geq \text{maks}\,\{x, y, x\}\}$

$= \{t \in \mathbb{N} \mid t \geq \text{maks}\,\{x, y\}\}$.

As $y \leq x$, we get $x \circ y \circ x = \{t \in \mathbb{N} \mid t \geq x\}$. This means that $x \in x \circ y \circ x$ for every $x \in \mathbb{N}$. Thus, every element $\mathbb{N}$ is a regular element. ■

Example 2.2

Let $H_1 = \{x, y, z, t\}$ is equipped with a hyperoperation $\circ$ defined in Table 2.1.

Table 2.1. Hyperoperation $\circ$ on H_1

$\circ$	x	y	z	t
x	$\{x\}$	$\{x,y\}$	$\{x,z\}$	$\{x,y,z,t\}$
y	$\{y\}$	$\{y\}$	$\{y,t\}$	$\{y,t\}$
z	$\{z\}$	$\{z,t\}$	$\{z\}$	$\{z,t\}$
t	$\{t\}$	$\{t\}$	$\{t\}$	$\{t\}$

From the table we can see the following properties.

1. The hyperoperation $\circ$ is a binary operation on H_1, that is for every $a, b \in H_1$ we have $a \circ b \in P^*(H_1)$. Here, $P^*(H_1)$ denotes the collection of nonempty subset of H_1.
2. The hyperoperation $\circ$ in H_1 is associative as $(a \circ b) \circ c = a \circ (b \circ c)$ for every $a, b, c \in H_1$.
3. Element *x*, *y*, *z*, and *t* are regular elements in $(H_1, \circ)$ as explained below. Since

$$
\begin{aligned}
x \circ H_1 \circ x &= \{x\} \circ \{x, y, z, t\} \circ \{x\} \\
&= [(x \circ x) \cup (x \circ y) \cup (x \circ z) \cup (x \circ t)] \circ \{x\} \\
&= [\{x\} \cup \{x, y\} \cup \{x, z\} \cup \{x, y, z, t\}] \circ \{x\} \\
&= \{x, y, z, t\} \circ \{x\} \\
&= (x \circ x) \cup (y \circ x) \cup (z \circ x) \cup (t \circ x) \\
&= \{x\} \cup \{y\} \cup \{z\} \cup \{t\} = \{x, y, z, t\},
\end{aligned}
$$

then $x \in x \circ H_1 \circ x$. This means that x is a regular element in $(H_1, \circ)$.

Since

$$y \circ H_1 \circ y = \{y\} \circ \{x, y, z, t\} \circ \{y\}$$
$$= [(y \circ x) \cup (y \circ y) \cup (y \circ z) \cup (y \circ t)] \circ \{y\}$$
$$= [\{y\} \cup \{y\} \cup \{y, t\} \cup \{y, t\}] \circ \{y\} = \{y, t\} \circ \{y\}$$
$$= (y \circ y) \cup (t \circ y) = \{y\} \cup \{t\} = \{y, t\},$$

then $y \in y \circ H_1 \circ y$. Therefore, y is a regular element in $(H_1, \circ)$.

We have

$$z \circ H_1 \circ z = \{z\} \circ \{x, y, z, t\} \circ \{z\}$$
$$= [(z \circ x) \cup (z \circ y) \cup (z \circ z) \cup (z \circ t)] \circ \{z\}$$
$$= [\{z\} \cup \{z, t\} \cup \{z\} \cup \{z, t\}] \circ \{z\} = \{z, t\} \circ \{z\}$$
$$= (z \circ z) \cup (t \circ z) = \{z\} \cup \{t\} = \{z, t\},$$

which implies that $z \in z \circ H_1 \circ z$. This shows us that z is a regular element in $(H_1, \circ)$.

As

$$t \circ H_1 \circ t = \{t\} \circ \{x, y, z, t\} \circ \{t\}$$
$$= [(t \circ x) \cup (t \circ y) \cup (t \circ z) \cup (t \circ t)] \circ \{t\}$$
$$= [\{t\} \cup \{t\} \cup \{t\} \cup \{t\}] \circ \{t\} = \{t\} \circ \{t\} = \{t\},$$

we find that $t \in t \circ H_1 \circ t$, i.e. t is a regular element in $(H_3, \circ)$.

Therefore, $(H_1, \circ)$ is a regular semihypergroup. ■

The second property we prove in this paper is the property involving the regular semihypergrup.

Theorem 2.2

If H is a regular semihypergroup and A is a subsemihypergroup of H, then A is a regular semihypergroup.

Proof. Since A is subsemihypergrup of H, then $A \subseteq H$; and according to Lemma 2.1, A is a semihypergrup. Now, if we take any $a \in A$, then $a \in H$. As H is a regular semihypergroup, then all elements of H are regular elements. As a result, the element a in A is also a regular element. This tell us that every element in A is a regular element, i.e. A is a regular semihypergroups. ■

We ilustrate Theorem 2.2 by looking at the following examples.

Example 2.3

In [2] page 50, it has been showed that the set $H_2 = \{a, b, c\}$ equipped with the hyperoperation given in Table 2.3 is a regular semihypergroup.

Table 2.2. Hyperoperation $\circ$ on H_2

$\circ$	a	b	c
a	$\{a\}$	$\{a, b\}$	$\{a, b, c\}$
b	$\{a\}$	$\{a, b\}$	$\{a, b, c\}$
c	$\{a\}$	$\{a, b\}$	$\{c\}$

Now, we will take $A \subseteq H_2$ in which $A = \{a, b\}$. Let A is equipped with the same hyperoperation given on H as depicted in the following table.

Table 2.3. Hyperoperation $\circ$ on A

$\circ$	a	b
a	$\{a\}$	$\{a, b\}$
b	$\{a\}$	$\{a, b\}$

Firstly, we will see that

$$A \circ A = \{a, b\} \circ \{a, b\} = (a \circ a) \cup (a \circ b) \cup (b \circ a) \cup (b \circ b)$$
$$= \{a\} \cup \{a, b\} \cup \{a\} \cup \{a, b\} = \{a, b\} = A.$$

Hence $A \circ A \subseteq A$, which implies that $(A, \circ)$ is a subsemihypergroup of $(H_1, \circ)$. Furthermore, we will find the following results.

$$a \circ A \circ a = \{a\} \circ \{a, b\} \circ \{a\} = [(a \circ a) \cup (a \circ b)] \circ \{a\}$$
$$= [\{a\} \cup \{a, b\}] \circ \{a\} = \{a, b\} \circ \{a\}$$
$$= (a \circ a) \cup (b \circ a) = \{a\} \cup \{a\} = \{a\},$$

so that $a \in a \circ A \circ a$, i.e. a is a regular element on $(A, \circ)$.

$$b \circ A \circ b = \{b\} \circ \{a, b\} \circ \{b\} = [(b \circ a) \cup (b \circ b)] \circ \{b\}$$
$$= [\{a\} \cup \{a, b\}] \circ \{b\} = \{a, b\} \circ \{b\}$$

$= (a \circ b) \cup (b \circ b) = \{a, b\} \cup \{a, b\} = \{a, b\}.$

so that $b \in b \circ A \circ b$, which show us that b is a regular element on $(A, \circ)$,

Since all the elements of A are regular elements, we prove that the subsemihypergroup $(A, \circ)$ of regular semihypergroup $(H_2, \circ)$ is also a regular semihypergroup. ■

Example 2.4

In Example 2.2, we found that $(H_1, \circ)$ is a regular semihypergroup. Now, let $B \subseteq H_1$ where $B = \{x, y\}$. The hyperoperation $\circ$ on B is provided in the following table.

Tabel 2.4. Hyperoperation $\circ$ on B

$\circ$	x	y
x	$\{x\}$	$\{x, y\}$
y	$\{y\}$	$\{y\}$

We have

$$B \circ B = \{x, y\} \circ \{x, y\} = (x \circ x) \cup (x \circ y) \cup (y \circ x) \cup (y \circ y)$$
$$= \{x\} \cup \{x, y\} \cup \{y\} \cup \{y\} = \{x, y\} = B,$$

which implies that $B \circ B \subseteq B$. This allow us to conclude that $(B, \circ)$ is a subsemihypergrup of $(H_1, \circ)$. We then examine all the elements of B to see whether the elements are regular.

$$x \circ B \circ x = \{x\} \circ \{x, y\} \circ \{x\} = [(x \circ x) \cup (x \circ y)] \circ \{x\}$$
$$= [\{x\} \cup \{x, y\}] \circ \{x\} = \{x, y\} \circ \{x\} = (x \circ x) \cup (y \circ x)$$
$$= \{x\} \cup \{y\} = \{x, y\},$$

so that $x \in x \circ B \circ x$; and hence x is a regular element on $(B, \circ)$.

$$y \circ B \circ y = \{y\} \circ \{x, y\} \circ \{y\} = [(y \circ x) \cup (y \circ y)] \circ \{y\}$$
$$= [\{y\} \cup \{y\}] \circ \{y\} = \{y\} \circ \{y\} = \{y\}$$

so that $y \in y \circ B \circ y$, which tells us that y a regular element on $(B, \circ)$. As all element in B are regular elements, we prove that the subsemihypergroup $(B, \circ)$ of regular semihypergroup $(H_2, \circ)$ is also a regular semihypergrup ■

The Cartesian product of two semihypergroups will result in a semihypergroup.

Lemma 2.3 (Davvaz, 2016: 49)

Let $(H,\circ)$ *and* $(H',\star)$ *are semihypergroups. The Cartesian product* $H \times H'$ *is endowed with hyperoperation* $\otimes$*:*

$$(x,y) \otimes (x',y') = \{(a,b) | a \in x \circ x', b \in y \star y'\}$$

for every $(x,y),(x',y') \in H \times H'$, *(*$H \times H'$*,* $\otimes$*) is semihypergroups.*

Proof. For any $(x,y),(x',y') \in H \times H'$, which means x, x' element of H and y, y' element of H'. Consequently, $x \circ x' \in P^*(H)$ and $y \star y' \in P^*(H')$. Thus, we get

$$(x,y) \otimes (x',y') = \{(a,b) | a \in x \circ x', b \in y \star y'\} \in P^*(H \times H').$$

This means that hyperoperation $\otimes$ on $H \times H'$ is a binary hyperoperation. Furthermore, for every $(x,y),(x',y'),(x'',y'') \in H \times H'$ satisfies

$$\begin{aligned}
[(x,y) \otimes (x',y')] \otimes (x'',y'') &= \{(a,b) | a \in x \circ x', b \in y \star y'\} \otimes \{(x'',y'')\} \\
&= \{(a,b) | a \in ((x \circ x') \circ x''), b \in ((y \star y') \star y'')\} \\
&= \{(a,b) | a \in (x \circ (x' \circ x'')), b \in (y \star (y' \star y''))\} \\
&= \{(x,y)\} \otimes \{(a,b) | a \in x' \circ x'', b \in y' \star y''\} \\
&= (x,y) \otimes [(x',y') \otimes (x'',y'')]
\end{aligned}$$

This shows that operation $\otimes$ on $H \times H'$ is associative. So, it is proved that $(H \times H', \otimes)$ is a semihypergroup. ■

The Cartesian product of two regular semihypergroups will result in a regular semihypergroup.

Theorem 2.4 (Davvaz, 2016: 50)

Let $(H, \circ)$ *and* $(H', \star)$ *are regular semihypergroups. If the Cartesian product* $H \times H'$ *is endowed with hyperoperation* $\otimes$:

$$(x,y) \otimes (x',y') = \{(a,b) | a \in x \circ x', b \in y \star y'\}$$

for every $(x,y),(x',y') \in H \times H'$, *then* $(H \times H', \otimes)$ *is a regular semihypergroup.*

Proof. It is known that $(H,\circ)$ and $(H',\star)$ are regular semihypergroups, respectively, so $(H,\circ)$ and $(H',\star)$ are semihypergroups. Thus, according to Lemma 2.3, the Cartesian product of two semihypergroups is a semihypergroup. Furthermore, for any $(x,y) \in H \times H'$, there is $(x',y') \in H \times H'$ such that

$$(x,y) \otimes (x',y') \otimes (x,y) \;= \{(a,b)|a \in x \circ x', b \in y \star y'\} \otimes \{(x,y)\}$$
$$= \{(a,b)\big|a \in \big((x \circ x') \circ x\big), b \in \big((y \star y') \star y\big)\}$$

To show that (x,y) is a regular element, we show that

$$(x,y) \in \{(a,b)\big|a \in \big((x \circ x') \circ x\big), b \in \big((y \star y') \star y\big)\}.$$

Since $(x,y) \in H \times H'$, then $x \in H$ and $y \in H'$ where H and H' are regular semihypergroups, respectively. This means that for every $x \in H$ there is $x' \in H$ such that $x \in x \circ x' \circ x$ and for every $y \in H'$ there is $y' \in H'$ such that $y \in y \star y' \star y$.

So, it is obatained

$$(x,y) \in \{(a,b)|a \in ((x \circ x') \circ x), b \in ((y \star y') \star y)\}$$

and consequently

$$(x,y) \in (x,y) \otimes (x',y') \otimes (x,y).$$

So, (x,y) is a regular element. Since (x,y) is any element in $H \times H'$, then all of elements in $H \times H'$ are regular elements. This proves that $(H \times H', \otimes)$ is a regular semihypergroup. ∎

Example 2.5 Let a set $H_1 = \{a,b,c\}$ is equipped with the following hyperoperation $\circ$.

Table 2.5 Hyperoperation $\circ$ on H_1

$\circ$	a	b	c
□	{□}	{□,□}	{□,□,□}
b	$\{a\}$	$\{a,b\}$	$\{a,b,c\}$
c	$\{a\}$	$\{a,b\}$	$\{c\}$

Table 2.6 Hiperoperation $\star$ on H_2

$\star$	p	q
p	$\{p\}$	$\{q\}$
q	$\{q\}$	$\{p,q\}$

Given also $H_2 = \{p, q\}$, which is equipped with the following hyperoperation $\star$. The Cartesian product of H_1 and H_2 is given by

$$H_1 \times H_2 = \{(a,p),(a,q),(b,p),(b,q),(c,p),(c,q)\}.$$

Hence, $(H_1 \times H_2, \otimes)$ is a regular semihypergroup. ■

As a consequence of Theorem 2.2, we have the following corollary.

Corollary 2.5

Let $(H, \circ)$ and $(H', \star)$ are regular semihypergroups. Let also $(A, \circ)$ and $(A', \star)$ are subsemihypergroup of $(H, \circ)$ and $(H', \star)$ respectively. If the Cartesian product $A \times A'$ is endowed with hyperoperation $\otimes$:

$$(x,y) \otimes (x',y') = \{(a,b) | a \in x \circ x', b \in y \star y'\}$$

for every $(x,y),(x',y') \in A \times A'$, then $(A \times A', \otimes)$ is a regular semihypergroup.

3. CONCLUSION

As a semigroup is a semihypergroup, some properties regarding the relation between semigroup and subsemigroup could be work for the relation between semihypergroup and subsemihypergroup. For example, we could investigate such properties for cyclic and simple semihypergroups.

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REFERENCES

[1] Bonansinga, P. and Corsini, P., *On Semihypergroup and Hypergroup Homomorphisms*, Unione Matematica Italiana. Bollettino. B. Serie VI, **1**(2) (1982), 717–727, Italian).

[2] Davvaz, B., *Semihypergroup Theory*, Academic Press, Cambridge, 2016

[3] Davvaz, B., *Some Results on Congruences on Semihypergroups*, Bulletin of the Malaysian Mathematical Sciences Society, Second Series, **23**(1) (2000), 53–58, 2000.

[4] Davvaz, B., Poursalavati, N.S., *Semihypergroups and S-hypersystems*, Pure Mathematics and Applications, **11** (2000), 43–49.

[5] Fasino, D., Freni, D., *Existence of Proper Semihypergroups of Type U on The Right*, Discrete Mathematics, **307**(22) (2007), 2826–2836.

[6] Guţan, C. *Simplifiable Semihypergroups,* in Algebraic Hyperstructures and Applications (Xánthi, 1990), 103–111, World Scientific Publishing, Teaneck, NJ, USA, 1991.

[7] Hasankhani, A. *Ideals in A Semihypergroup and Green's Relations*, Ratio Mathematica, no. 13, 29–36, 1999.

[8] Javarpour, M., Leoreanu-Fotea, V. , Zolfaghari, A., *Weak complete parts in semyhypergroups*, Iranian Journal of Mathematical Secience and Informatics, **8**(2) (2013), 101 – 109.

[9] Leoreanu, V., *About the simplifiable cyclic semihypergroups,* Italian Journal of Pure and Applied Mathematics, **7** (2000), 69–76.

[10] Marty, F., *Sur Une Generalization de la Notion de Groupe*, in Proceedings of the 8th Congress des Mathematiciens Scandinaves, Stockholm, Sweden, 45–49, 1934.

[11] Onipchuk, S.V., *Regular semihypergroups*, Matematicheskiĭ Sbornik, **183**(6) (1992), 43–54, 1992 (Russian), translation in Russian Academy of Sciences. Sbornik Mathematics, **76**(1993), 155–164.