Jurnal Ilmiah Matematika dan Pendidikan Matematika (JMP)
Vol. 13 No. 1, Juni 2021, hal. 15-26
ISSN (Cetak) : 2085-1456; ISSN (*Online*) : 2550-0422

SOLVING THE INVERSE KINEMATICAL PROBLEM OF A ROBOT ARM BY USING GROEBNER BASIS

Alisya Masturoh
Department of Mathematics,Jenderal Soedirman University
paddraalisyamasturoh@gmail.com

Bambang Hendriya Guswanto
Department of Mathematics,Jenderal Soedirman University
bambang.guswanto@unsoed,ac.id

Triyani
Department of Mathematics,Jenderal Soedirman University
triyani@unsoed.ac.id

Abstract. *The inverse kinematical problem of a robot arm is a problem to find some appropriate joint configurations for a pair of position and direction of a robot hand which is represented by a polynomial equations system. The system is solved by employing Groebner basis notion. Here, the appropriate joint configurations for a pair of position and direction of the hand of a robot with three segments are obtained.*

Keywords : *inverse kinematical problem, robort am, Groebner basis*

Abstrak. Masalah kinematika invers lengan robot adalah suatu masalah menentukan konfigurasi bersama yang sesuai untuk pasangan posisi dan arah tangan robot yang dinyatakan oleh suatu sistem polynomial. Sistem ini diselesaikan dengan menggunakan konsep basis Groebner. Di sini, konfigurasi bersama yang sesuai untuk pasangan posisi dan arah tangan robot dengan tiga segmen diperoleh.

Kata kunci : masalah kinematika inverse, lengan robot, basis Groebner

1. INTRODUCTION

Groebner basis of a polynomial ideal is a spanning subset of the ideal causing the leading term of any non-zero element of the ideal can be divided by one or more leading terms of the elements of Groebner basis. Groebner basis is employed to solve polynomial equations system (see [1,2]). This notion can be applied to robotics problems such as inverse kinematical problems. Inverse kinematical problem is a problem to find some appropriate joint configurations for a pair of position and direction of a robot hand. In [3], Natarajan et al discuss kinematical problem of a robot arm with three fingers using graphical method. Kendricks [4] used the Denavit-Hartenberg matrix and Groebner basis to solve the inverse kinematical problem of the GMF Robotics A-510 robot and then compared both methods.

This paper studies how Groebner basis is used to solve an inverse kinematical problem for a robot arm with three segments. We obtain all combinations of joints settings satisfying certain position and direction of the robot hand which solves the polynomial equations system of the reduced Groebner basis reperesenting the inverse kinematical problem.

This paper is composed of four sections. In the second section, a method to solve the inverse kinematical problem is given. In the third section, the solvability of the polynomial equations system of the reduced Groebner basis representing the inverse kinematical problem for the robot arm with three segments is discussed. In conclusion, all combinations of joints settings satisfying certain position and direction of the robot hand are mentioned.

2. METHODS

To solve the inverse kinematical problem, we perfom the following steps :

1. forming an ideal generated by functions which represents the position and direction of the robot hand;
2. forming Groebner basis for the ideal in step 1;
3. forming the reduced Groebner Basis of the Groebner basis obtained in step 2;
4. determining all combinations of joints settings satisfying certain position and direction of the robot hand solving the polynomial equations system of the reduced Groebner basis.

3. RESULTS AND DISCUSSION

We here consider a robot arm with n segments with the position of the robot hand in a global coordinate system is (a,b) and the direction of the robot hand is $\theta = \theta_1 + \theta_2 + \cdots + \theta_{n-1}$. The value of $\theta_1, \theta_2, \cdots, \theta_{n-1}$ will be first determined. Based on the forward kinematical problem solving (see [5]),

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \sum_{i=2}^{n} l_i \cos\left(\theta_1 + \cdots + \theta_{i-1}\right) \\ \sum_{i=2}^{n} l_i \sin\left(\theta_1 + \cdots + \theta_{i-1}\right) \end{pmatrix}.$$

If the robot arm has three segments then

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} l_2 \cos\theta_1 + l_3 \cos\left(\theta_1 + \theta_2\right) \\ l_2 \sin\theta_1 + l_3 \sin\left(\theta_1 + \theta_2\right) \end{pmatrix}.$$

Note that

$$\cos\left(\theta_1 + \theta_2\right) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2,$$

and

$$\sin\left(\theta_1 + \theta_2\right) = \cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2.$$

If $c_i = \cos\theta_i$ dan $s_i = \sin\theta_i$ then

$$\cos\left(\theta_1 + \theta_2\right) = c_1 c_2 - s_1 s_2 \text{ and } \sin\left(\theta_1 + \theta_2\right) = c_1 s_2 + s_1 c_2.$$

Suppose that variables c_2, s_2, c_1, s_1 is ordered by *lexiographic* order with $c_2 > s_2 > c_1 > s_1$. Then

$$f_1 = l_3 c_2 c_1 - l_3 s_2 s_1 + l_2 c_1 - a \text{ and } f_2 = l_3 c_2 s_1 + l_3 s_2 c_1 + l_2 s_1 - b.$$

An Arm with three segments has two revolute joints

$$f_3 = c_1^2 + s_1^2 - 1 \text{ and } f_4 = c_2^2 + s_2^2 - 1.$$

To solve this system, we first compute the Groebner basis of the system using lex order. The value for $c_2, s_2, c_1,$ and s_1 will be determined using the Groebner basis. First, we form $\boldsymbol{I} = \langle f_1, f_2, f_3, f_4 \rangle$ which is an ideal generated by $f_1, f_2, f_3,$ and f_4. Then, by Buncberger algorithm, Groebner basis for ideal $\boldsymbol{I}$ can be obtained. We find that Groebner basis for $\boldsymbol{I} = \langle f_1, f_2, f_3, f_4 \rangle$ is $G = \{ f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9 \}$ where

$$f_1 = l_3 c_2 c_1 - l_3 s_2 s_1 + l_2 c_1 - a,$$

$$f_2 = l_3 c_2 s_1 + l_3 s_2 c_1 + l_2 s_1 - b,$$

$$f_3 = c_1^2 + s_1^2 - 1,$$

$$f_4 = c_2^{\ 2} + s_2^{\ 2} - 1,$$

$$f_5 = -s_2 + \frac{b}{l_3}c_1 - \frac{a}{l_3}s_1,$$

$$f_6 = c_2 - \frac{a}{l_3}c_1 - \frac{b}{l_3}s_1 + \frac{l_2}{l_3},$$

$$f_7 = c_1 s_1 - \frac{(a^2 + b^2 - l_3^{\ 2} + l_2^{\ 2})}{2bl_2}c_1 - \frac{a}{b}s_1^{\ 2} + \frac{a}{b},$$

$$f_8 = c_1 + \frac{2l_2(a^2+b^2)}{a(a^2+b^2-l_3^{\ 2}+l_2^{\ 2})}s_1^{\ 2} - \frac{b}{a}s_1 - \frac{2al_2}{a^2+b^2-l_3^{\ 2}+l_2^{\ 2}},$$

$$f_9 = -s_1^{\ 3} + \frac{b(a^2+b^2-l_3^{\ 2}+l_2^{\ 2})}{l_2(a^2+b^2)}s_1^{\ 2} + \frac{4a^2l_2^{\ 2} - (a^2+b^2-l_3^{\ 2}+l_2^{\ 2})^2}{4l_2^{\ 2}(a^2+b^2)}$$

and

$$f_{10} = s_1^{\ 2} - \frac{b(a^2+b^2-l_3^{\ 2}+l_2^{\ 2})}{l_2(a^2+b^2)}s_1 + \frac{(a^2+b^2-l_3^{\ 2}+l_2^{\ 2})^2 - 4a^2l_2^{\ 2}}{4l_2^{\ 2}(a^2+b^2)}.$$

Then, the reduced Groebner basis is

$$G = \{g_1, g_2, g_{3,} g_4\} \qquad (1)$$

where

$$g_1 = c_2 - \frac{a^2+b^2-l_3^{\ 2}-l_2^{\ 2}}{2l_3l_2},$$

$$g_2 = s_2 + \frac{a^2+b^2}{al_3}s_1 - \frac{b(a^2+b^2-l_3^{\ 2}+l_2^{\ 2})}{2al_2l_3},$$

$$g_{3,} = c_1 + \frac{b}{a}s_1 - \frac{a^2+b^2-l_3^{\ 2}+l_2^{\ 2}}{2al_2},$$

and

$$g_4 = s_1^{\ 2} - \frac{b(a^2+b^2-l_3^{\ 2}+l_2^{\ 2})}{l_2(a^2+b^2)}s_1 + \frac{(a^2+b^2-l_3^{\ 2}+l_2^{\ 2})^2 - 4a^2l_2^{\ 2}}{4l_2^{\ 2}(a^2+b^2)}.$$

Based on Groebner basis (1), we find that

$$l_2 \neq 0, l_3 \neq 0, a \neq 0 \text{ and } a^2 + b^2 \neq 0 .$$

The last polynomial in (1) is in quadratic form. Then, there are at most two solutions for this polynomial. If $l_2 \neq l_3$ then

$$s_1 = \frac{b(a^2+b^2-l_3^2+l_2^2)}{2l_2(a^2+b^2)} \pm \frac{|a|\sqrt{2(l_2^2+l_3^2)(a^2+b^2)-(a^2+b^2)^2-(l_2^2-l_3^2)^2}}{2l_2(a^2+b^2)} \quad (2)$$

which exists if

$$0 < (a^2+b^2)^2+(l_2^2-l_3^2)^2 \le 2(l_2^2+l_3^2)(a^2+b^2).$$

If $(a^2+b^2)^2+(l_2^2-l_3^2)^2 = 2(l_2^2+l_3^2)(a^2+b^2)$, there is one solution

$$s_1 = \frac{b(a^2+b^2-l_3^2+l_2^2)}{2l_2(a^2+b^2)}.$$

If $s_1 = \frac{b(a^2+b^2-l_3^2+l_2^2)}{2l_2(a^2+b^2)}$ then

$$s_2 = 0,\ c_1 = \frac{a(a^2+b^2-l_3^2+l_2^2)}{2l_2(a^2+b^2)},\ \text{and}\ c_2 = \frac{a^2+b^2-l_3^2-l_2^2}{2l_3 l_2}$$

The value $s_2 = \sin\theta_2 = 0$ implies $\theta_2 = 0$ or $\theta_2 = \pi$. Then, $c_2 = 1$. Thus, there is only one pair of joint $(\theta_1, \theta_2) = (\theta, 0)$.

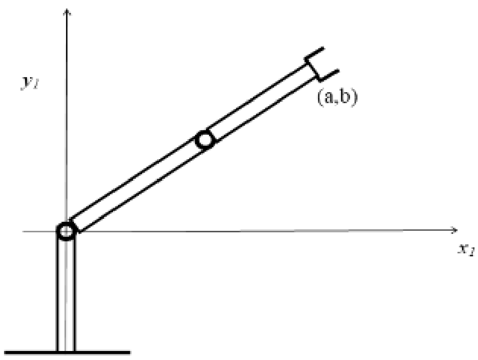

Figure 1. The joint setting for $(a^2+b^2)^2+(l_2^2-l_3^2)^2 = 2(l_2^2+l_3^2)(a^2+b^2)$.

Next, for $(a^2+b^2)^2+(l_2^2-l_3^2)^2 < 2(l_2^2+l_3^2)(a^2+b^2)$, there are two values of s_1 satisfying this case. If s_1 is given then the value of s_2 and c_1 can be determined by substituting s_1 into g_2 and g_3. Note that there are two values of s_1 for certain (a,b). Consequently, there are also two values of s_2 and c_1, those are

$$s_2 = \pm \frac{\sqrt{2(l_2^2 + l_3^2)(a^2 + b^2) - (a^2 + b^2)^2 - (l_2^2 - l_3^2)^2}}{2l_2 l_3}$$

and

$$c_1 = \frac{\mathrm{a}(a^2 + b^2 - l_3^2 + l_2^2)}{2l_2(a^2 + b^2)} \pm \frac{b\sqrt{2(l_2^2 + l_3^2)(a^2 + b^2) - (a^2 + b^2)^2 - (l_2^2 - l_3^2)^2}}{2l_2(a^2 + b^2)}$$

Then,

$$c_2 = \frac{a^2 + b^2 - l_3^2 - l_2^2}{2l_3 l_2}.$$

For $(a^2 + b^2)^2 + (l_2^2 - l_3^2)^2 \le 2(l_2^2 + l_3^2)(a^2 + b^2)$, there are two pairs of joint satisfying this case.

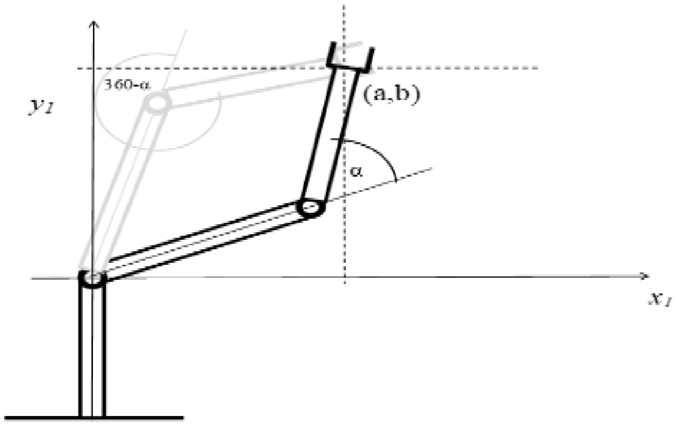

Figure 2. The joint setting for $(a^2 + b^2)^2 + (l_2^2 - l_3^2)^2 \le 2(l_2^2 + l_3^2)(a^2 + b^2)$.

If $l_2 = l_3$ then

$$s_1 = \frac{b}{2l_2} \pm \frac{|a|\sqrt{4l_2^2(a^2 + b^2) - (a^2 + b^2)^2}}{2l_2(a^2 + b^2)} \tag{3}$$

exists if $0 < a^2 + b^2 \le 4l_2^2$. If $a^2 + b^2 = 4l_2^2$ then $s_1 = \frac{b}{2l_2}$. Subtituting $s_1 = \frac{b}{2l_2}$ into g_2 and g_3, we find that

$$s_2 = 0,\ c_1 = \frac{a}{2l_2}, \text{ and } c_2 = 1.$$

Then, $c_2 = \cos\theta_2 = 1$ and $s_2 = \sin\theta_2 = 0$ imply $\theta_2 = 0$. Thus, the joint setting is $\theta_2 = 0$ and $\theta_1 = \theta$ with $\cos\theta = \cos\theta_1 = \dfrac{a}{2l_2}$ and $\sin\theta = \sin\theta_1 = \dfrac{b}{2l_2}$. This case is geometrically confirmed since if $l_2 = l_3$ then the longest distance between the center of global coordinate system and the hand tip is $l_2 + l_3 = 2l_2$. Consequently, for $a^2 + b^2 = 4l_2^2$, the joint setting for θ_2 is 0.

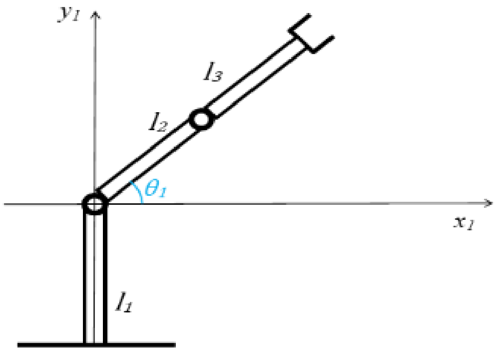

Figure 3. The joint setting for $a^2 + b^2 = 4l_2^2$.

There is only one solution for $l_2 = l_3$ with $a^2 + b^2 = 4l_2^2$.

If $a^2 + b^2 < 4l_2^2$ then

$$s_1 = \frac{b}{2l_2} \pm \frac{|a|\sqrt{4l_2^2(a^2+b^2)-(a^2+b^2)^2}}{2l_2(a^2+b^2)}$$

$$s_2 = \pm\frac{\sqrt{4l_2^2(a^2+b^2)-(a^2+b^2)^2}}{2l_2^2}$$

$$c_1 = \frac{a}{2l_2} \pm \frac{b\sqrt{4l_2^2(a^2+b^2)-(a^2+b^2)^2}}{2l_2(a^2+b^2)}$$

Then, there are two pairs of joints. In fact, the hand tip position can be placed coincidentally on y_1-axis that means $a = 0$. If we subtitute $a = 0$ into the Groebner basis then the solution is undefined mathematically. But, the solution for this case is geometrically defined. To solve this problem, we recall the ideal $\boldsymbol{I} = \langle f_1, f_2, f_3, f_4 \rangle$ and subtitute $a = 0$ into

$$f_1 = l_3 c_2 c_1 - l_3 s_2 s_1 + l_2 c_1,$$

$$f_2 = l_3 c_2 s_1 + l_3 s_2 c_1 + l_2 s_1 - b,$$

$$f_3 = c_1^{\ 2} + s_1^{\ 2} - 1,$$

and

$$f_4 = c_2^{\ 2} + s_2^{\ 2} - 1.$$

Then, we find the reduced Groebner basis for the ideal, that is

$$G = \{g_1, g_2, g_{3,} g_4\} \tag{4}$$

with

$$g_1 = \ f_6 = c_2 - \frac{b^2 - (l_2^{\ 2} + l_3^{\ 2})}{2 l_3 l_2},$$

$$g_2 = \ f_5 = s_2 - \frac{b}{l_3} c_1,$$

$$g_{3,} = \ f_3 = c_1^{\ 2} + \frac{(b^2 + l_2^{\ 2} - l_3^{\ 2})^2 - 4b^2 l_2^{\ 2}}{4b^2 l_2^{\ 2}},$$

and

$$g_4 = \ f_9 = s_1 - \frac{b^2 + l_2^{\ 2} - l_3^{\ 2}}{2b l_2}.$$

Polynomial g_3 is a quadratic formula in c_1, so there are at most two solutions for c_1. If $l_2 \neq l_3$ then

$$c_1 = \pm \frac{\sqrt{(2bl_2)^2 - (b^2 + l_2^{\ 2} - l_3^{\ 2})^2}}{2|b| l_2} \tag{5}$$

which exist if $0 \le (b^2 + l_2^{\ 2} - l_3^{\ 2})^2 \le (2bl_2)^2$. If $(b^2 + l_2^{\ 2} - l_3^{\ 2})^2 = (2bl_2)^2$ then

$$c_1 = 0, c_2 = 1, s_1 = 1, s_2 = 0.$$

Thus, there is only one setting for $a = 0$ and $(b^2 + l_2^{\ 2} - l_3^{\ 2})^2 = (2bl_2)^2$, that is

$$\theta_2 = 0, \theta_1 = \frac{1}{2}\pi.$$

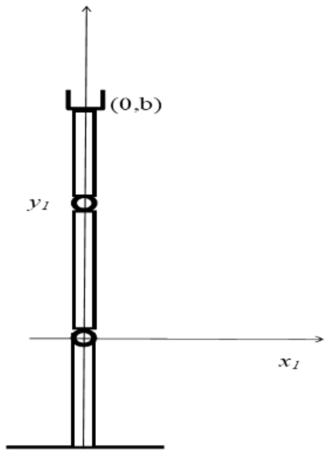

Figure 4. The joint setting for $0 \le (b^2 + l_2{}^2 - l_3{}^2)^2 \le (2bl_2)^2$.

If $(b^2 + l_2{}^2 - l_3{}^2)^2 < (2bl_2)^2$, there are two solutions for θ_1 and θ_2.

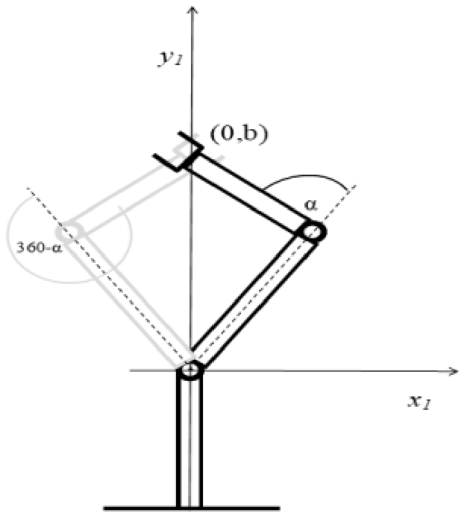

Figure 5. The joint setting for $(b^2 + l_2{}^2 - l_3{}^2)^2 < (2bl_2)^2$.

Consider the hand tip placed at the center of the global coordinate system or $(a, b) = (0,0)$. This case exists only if the second segment has the same length as the third segment. In this case, the solution is $\theta_2 = \pi$ and $\theta_1 = \theta - \pi$.

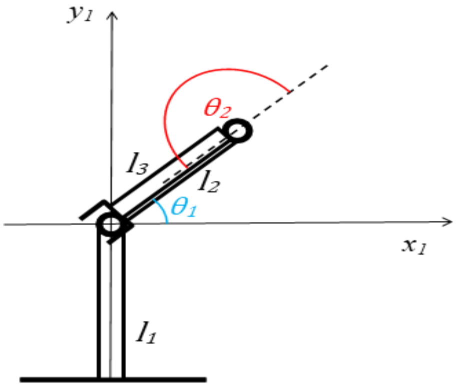

Figure 6. The joint setting for $(a, b) = (0,0)$.

4. CONCLUSION

Based on the discussion in Section 3, we conclude that:

1. for $l_2 \neq l_3$, if $(a^2+b^2)^2+(l_2^2-l_3^2)^2 \leq 2(l_2^2+l_3^2)(a^2+b^2)$, there are two joint settings;
2. for $l_2 = l_3$, if $a^2+b^2 = 4l_2^2$, there is only one joint setting;
3. for $l_2 = l_3$, if $a^2+b^2 < 4l_2^2$, there are two joint settings;
4. for $l_2 \neq l_3$ and $a=0$, if $(b^2+l_2^2-l_3^2)^2 = (2bl_2)^2$, there is only one joint setting;
5. for $l_2 \neq l_3$ and $a=0$, if $(b^2+l_2^2-l_3^2)^2 < (2bl_2)^2$, there are two joint settings;
6. the position $(a,b)=(0,0)$ exists if and only if $l_2 = l_3$ and there are unlimited joint settings with $\theta_2 = \pi$;
7. there is no joint setting for $(a^2+b^2)^2+(l_2^2-l_3^2)^2 > 2(l_2^2+l_3^2)(a^2+b^2)$ with $l_2 \neq l_3$, $a^2+b^2 > 4l_2^2$ with $l_2 = l_3$, and $(b^2+l_2^2-l_3^2)^2 > (2bl_2)^2$ with $l_2 \neq l_3$ and $a=0$.

REFERENCES

[1] Cox, D., Little, J., and O'Shea, D., *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Comutative Algebra*, Springer, New York, 2007.

[2] Haset, B., *Introduction to Algebraic Geometry*, Cambridge University Press, 2007.

[3] Natarajan, E. and Dhar, L., *Kinematic Analysis of Three Fingered Robot Hand Using Graphical Method*, International Journal of Engineering and Technology, **5** (2013), 528-22

[4] Kendricks, K., *Solving the Inverse Kinematic Robotic Problem: A Compparison Study of The Denavit-Hartenberg Matrix and Groebner Basis Theory*, Auburn University, 2007.

[5] Guswanto, B. H., Masturo, A., and Triyani, *Pemecahan Masalah Kinematika ke Depan pada Tangan Robot n-Segmen*, Jurnal Ilmiah Matematika dan Pendidikan Matematika **12** (2) (2020), 1-13.