JMP : Volume 8 Nomor 1, Juni 2016, hal 1-7

A WEAK-(p, q) INEQUALITY FOR FRACTIONAL INTEGRAL OPERATOR ON MORREY SPACES OVER METRIC MEASURE SPACES

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Abstract. This paper presents a weak-(p, q) inequality for fractional integral operator on Morrey spaces over metric measure spaces of nonhomogeneous type. Both parameters p and q are greater than or equal to one. The weak-(p, q) inequality is proved by employing an inequality involving maximal operator on the spaces under consideration.

Keywords: Fractional integral operator, maximal operator,Morrey spaces, nonhomogeneous, weak type inequality.

1. Introduction

Fractional integral operator, which is firstly defined by (Hardy and Littlewood, 1927), is an inverse for a power of the Laplacian operator on Euclidean spaces. This operator is also called the Riesz potential. Riesz (1938) introduced the potential as an extension of the Newtonian potential. Nowadays, it can be found that the works on fractional integral operator have been developed in some directions. For examples, (Adams, 1975), (Nakai, 1994), (Kurata, et al., 2002), (Gunawan and Shwaningrum, 2013) and (Eridani, et al., 2014) provided some results on homogeneous type spaces; meanwhile (García-Cuerva and Martell, 2001), (Liu and Shu, 2011), (Sihwaningrum, et al., 2012), and (Sawano an Shimamura, 2013) provided some results on nonhomogeneous type spaces. A metric measure space (X, d, μ) is a homogeneous type space if the Borel measure μ satisfies the doubling condition, that is there exists a positive constant C such that for every ball $B(a, r)$ the condition

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)) \tag{1}$$

holds. In equation (1), a is the center of the ball and r is the radius of the ball. If the doubling condition does not hold, then we have a metric measure space of nonhomogeneous type. In spaces of nonhomogeneous type, the doubling condition can be replaced by the growth condition

$$\mu(B(x, r)) \leq C\, r^n \tag{2}$$

where n is less than or equal to the dimension of the metric measure spaces. The action on the spaces of nonhomogeneous type goes back to the works of (Nazarov, et al., 1998), (Tolsa, 1998), and (Verdera, 2002).

In the metric measure spaces of nonhomogeneous type, (García-Cuerva and Gatto, 2004) defined the fractional integral operator I_α to be

$$I_\alpha f(x) := \int_X \frac{f(y)}{d(x,y)^{n-\alpha}}\, d\mu(y). \tag{3}$$

The fractional integral operator (equation (3)) has been proved to satisfy the weak-$(1,q)$ an inequality in Lebesgue spaces over metric measure spaces of nonhomogeneous type (García-Cuerva and Gatto, 2004), in Morrey spaces over metric measure of non homogenenous type (Sihwaningrum, 2016) and in generalized Morrey spaces over metric measure spaces of non homogeneous type (Sihwaningrum, et al., 2015). This kind of weak type is also established in (Sihwaningrum and Sawano, 2013) for another version of fractional integral operator. The weak inequalities measure the size of the distribution function (Duoandikoetxea, 2001). As (Hakim and Gunawan, 2013) had a result on the weak-(p,q) inequality (where $1 \le p \le q < \infty$) for fractional integral operator in the generalized Morrey spaces over Euclidean spaces of nonhomogeneous type, the results on the weak-$(1,q)$ inequality on Morrey spaces over metric measure spaces of nonhomogeneous type will extended in this paper into a weak-(p,q) inequality. Morrey spaces were first introduced by (Morrey, 1940); and in this paper, Morrey spaces over metric spaces of non homogeneous type $L^{p,\lambda}(\mu) := L^{p,\lambda}(X,\mu)$ (for $1 \le p < \infty$ and $0 \le \lambda < n$) contain all functions in L^p_{loc} in which

$$\|f\|^p_{L^{p,\lambda}(\mu)} := \sup_{B(x,r)} \left(\frac{1}{(\mu(B(x,2r)))^\lambda} \int_{B(x,r)} |f(y)|^p \, d\mu(y) \right)^{1/p} < \infty.$$

2. Main Results

To prove the weak-(p,q) inequality of fractional integral operator on Morrey spaces over metric measure spaces of nonhomogeneous type, we need the following inequality which involves maximal operator M. This operator is defined by

$$Mf(x) := \sup_{r>0} \frac{1}{\mu(B(x,2r))} \int_{B(x,r)} |f(y)|\, d\mu(y) \quad (x \in \operatorname{supp}(\mu)) \tag{4}$$

(Sawano, 2005). Some properties of maximal operator M can be found for example in (Terasawa, 2006). Note that from now on, C denotes different positive constants.

Theorem 2.1. *Let $\chi_{B(a,r)}$ be the characteristic function of ball $B(a,r)$. If $f \in L^{p,\lambda}(\mu)$ for $1 \le p < \infty$ and $0 < \lambda < 1$, then for any ball $B(a,r)$ on X we have*

$$\int_X |f(y)|^p M\chi_{B(a,r)}\, d\mu(y) \le C r^{n\lambda} \|f\|^p_{L^{p,\lambda}(\mu)}.$$

Proof. By applying the growth condition in equation (2), for any function f in $L^{p,\lambda}(\mu)$ we get

$$\begin{aligned}
&\int_X |f(y)|^p M\chi_{B(a,r)}\, d\mu(y)\\
&\quad\leq \int_{B(a,r)} |f(y)|^p M\chi_{B(a,r)}\, d\mu(y) + \sum_{j=1}^{\infty} \int_{B(a,2^{j+1}r)\setminus B(a,2^j r)} |f(y)|^p M\chi_{B(a,r)}\, d\mu(y)\\
&\quad\leq C\left(\int_{B(a,r)} |f(y)|^p\, d\mu(y) + \sum_{j=1}^{\infty} 2^{-jn} \int_{B(a,2^{j+1}r)\setminus B(a,2^j r)} |f(y)|^p\, d\mu(y)\right)\\
&\quad\leq C\|f\|^p_{L^{p,\lambda}(\mu)}\left(\mu(B(a,2r))^\lambda + \sum_{j=1}^{\infty} 2^{-jn}\mu(B(a,2^{j+2}r))^\lambda\right)\\
&\quad\leq C\|f\|^p_{L^{p,\lambda}(\mu)}\left((2r)^{n\lambda} + \sum_{j=1}^{\infty} 2^{-jn}(2^{j+2}r)^{n\lambda}\right)\\
&\quad= Cr^{n\lambda}\|f\|^p_{L^{p,\lambda}(\mu)}\left(1 + \sum_{j=1}^{\infty} 2^{-jn(1-\lambda)}\right).
\end{aligned}$$

Since $1-\lambda > 0$, then $\sum_{j=1}^{\infty} 2^{-jn(1-\lambda)}$ is convergent. As a result,

$$\int_X |f(y)|^p M\chi_{B(a,r)}\, d\mu(y) \leq Cr^{n\lambda}\|f\|^p_{L^{p,\lambda}(\mu)}.$$

Therefore, the proof is complete. □

Having Theorem 2.1, we are now able to get the weak-(p,q) inequality for fractional integral operator I_α.

Theorem 2.2. *Let* $0 < \alpha < n$, $1 \leq p \leq q < \infty$, *and* $0 \leq \lambda < 1 - \frac{\alpha p}{n}$. *If* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n(1-\lambda)}$, *then*

$$\mu\left(\{x \in B(a,r) : |I_\alpha f(x) > \gamma\}\right) \leq Cr^{n\lambda}\left(\frac{\|f\|_{L^{p,\lambda}(\mu)}}{\gamma}\right)^q.$$

Proof. For any positive R, we can write $|I_\alpha f(x)|$ as

$$\begin{aligned}
|I_\alpha f(x)| &\leq \int_{B(x,R)} \frac{|f(y)|}{d(x,y)^{n-\alpha}}\, d\mu(y) + \int_{X\setminus B(x,R)} \frac{|f(y)|}{d(x,y)^{n-\alpha}}\, d\mu(y)\\
&= A_1 + A_2.
\end{aligned}$$

An estimate for A_2 is

$$\begin{aligned}
A_2 &\le \int_{d(x,y)\ge R} \frac{|f(y)|}{d(x,y)^{n-\alpha}}\, d\mu(y) \\
&\le \sum_{j=0}^{\infty} \int_{B(x,2^{j+1}R)\setminus B(x,2^jR)} \frac{|f(y)|}{d(x,y)^{n-\alpha}}\, d\mu(y) \\
&\le \sum_{j=0}^{\infty} \frac{1}{(2^jR)^{n-\alpha}} \int_{B(x,2^{j+1}R)} |f(y)|\, d\mu(y) \\
&= \sum_{j=0}^{\infty} \frac{2^{2n}(2^jR)^{\alpha}}{(2^{j+2}R)^n} \int_{B(x,2^{j+1}R)} |f(y)|\, d\mu(y).
\end{aligned}$$

Since μ satisfies the growth measure condition, then

$$\begin{aligned}
A_2 &\le \sum_{j=0}^{\infty} \frac{R^{\alpha}2^{j\alpha}}{\mu(B(x,2^{j+2}R))} \int_{B(x,2^{j+1}R)} |f(y)|\, d\mu(y) \\
&\le CR^{\alpha} \sum_{j=0}^{\infty} \frac{2^{j\alpha}}{\mu(B(x,2^{j+2}R))} \left(\int_{B(x,2^{j+1}R)} |f(y)|^p\, d\mu(y) \right)^{1/p} \left(\int_{B(x,2^{j+1}R)} d\mu(y) \right)^{1-1/p} \\
&= CR^{\alpha} \sum_{j=0}^{\infty} \frac{2^{j\alpha}\mu(B(x,2^{j+1}R))^{1-1/p}}{\mu(B(x,2^{j+2}R))^1 - \lambda/p} \left(\frac{1}{\mu(B(x,2^{j+2}R))^{\lambda}} \int_{B(x,2^{j+1}R)} |f(y)|^p\, d\mu(y) \right)^{1/p} \\
&\le CR^{\alpha}\|f\|_{L^{p,\lambda}(\mu)} \sum_{j=0}^{\infty} \left(2^{j+1}R\right)^{n(1-1/p)} \left(2^{j+2}R\right)^{n(\lambda/p-1} \\
&\le CR^{\alpha+n(\lambda/p-1/p)}\|f\|_{L^{p,\lambda}(\mu)} \sum_{j=0}^{\infty} 2^{j(\alpha+n(\lambda/p-1/p))}.
\end{aligned}$$

By using the assumption $0 \le \lambda < 1 - \frac{\alpha p}{n}$, we find that $\sum_{j=0}^{\infty} 2^{j(\alpha+n(\lambda/p-1/p))}$ is convergent; and hence

$$A_2 \le CR^{\alpha+\frac{n}{p}(\lambda-1)}\|f\|_{L^{p,\lambda}(\mu)}.$$

Now, an estimate for A_1 is given by

$$\begin{aligned}
A_1 &\le \left(\int_{B(x,R)} \frac{|f(y)|^p}{d(x,y)^{n-\alpha}}\, d\mu(y) \right)^{1/p} \left(\int_{B(x,R)} \frac{|f(y)|}{d(x,y)^{n-\alpha}}\, d\mu(y) \right)^{1-1/p} \\
&\le \left(\int_{B(x,R)} \frac{|f(y)|^p}{d(x,y)^{n-\alpha}}\, d\mu(y) \right)^{1/p} \left(\frac{\mu(B(x,R))}{R^{n-\alpha}} \right)^{1-1/p} \\
&\le \left(\int_{B(x,R)} \frac{|f(y)|^p}{d(x,y)^{n-\alpha}}\, d\mu(y) \right)^{1/p} \left(\frac{CR^n}{R^{n-\alpha}} \right)^{1-1/p} \\
&\le CR^{\alpha(1-1/p)} \left(\int_{B(x,R)} \frac{|f(y)|^p}{d(x,y)^{n-\alpha}}\, d\mu(y) \right)^{1/p}.
\end{aligned}$$

If we let

$$CR^{\alpha+\frac{n}{p}(\lambda-1)}\|f\|_{L^{p,\lambda}(\mu)} = \frac{\gamma}{2},$$

then $A_2 \leq \frac{\gamma}{2}$ and $\left\{x \in B(a,r) : A_2 > \frac{\gamma}{2}\right\} = \emptyset$. As a result,

$$\begin{aligned}
&\mu\left(\{x \in B(a,r) : |I_\alpha f(x)| > \gamma\}\right) \\
&\quad \leq \mu\left(\left\{x \in B(a,r) : A_1 > \frac{\gamma}{2}\right\}\right) + \mu\left(\left\{x \in B(a,r) : A_2 > \frac{\gamma}{2}\right\}\right) \\
&\quad = \mu\left(\left\{x \in B(a,r) : A_1 > \frac{\gamma}{2}\right\}\right) \\
&\quad = \mu\left(\left\{x \in B(a,r) : CR^{\alpha(1-1/p)}\left(\int_{B(x,R)} \frac{|f(y)|^p}{d(x,y)^{n-\alpha}}\, d\mu(y)\right)^{1/p} > \frac{\gamma}{2}\right\}\right) \\
&\quad = \mu\left(\left\{x \in B(a,r) : \int_{B(x,R)} \frac{|f(y)|^p}{d(x,y)^{n-\alpha}}\, d\mu(y) > \left(\frac{\gamma}{2CR^{\alpha(1-1/p)}}\right)^p\right\}\right) \\
&\quad \leq \frac{C\left(R^{\alpha(1-1/p)}\right)^p}{\gamma^p} \int_{B(a,r)} \int_{B(x,R)} \frac{|f(y)|^p}{d(x,y)^{n-\alpha}}\, d\mu(y)\, d\mu(x) \\
&\quad = \frac{C\left(R^{\alpha(1-1/p)}\right)^p}{\gamma^p} \int_X \int_{B(x,R)} \frac{|f(y)|^p \chi_{B(a,r)}(x)}{d(x,y)^{n-\alpha}}\, d\mu(y)\, d\mu(x) \\
&\quad \leq \frac{CR^{\alpha(p-1)}}{\gamma^p} \int_X |f(y)|^p \left(\int_{B(y,R)} \frac{\chi_{B(a,r)}(x)}{d(x,y)^{n-\alpha}}\, d\mu(x)\right) d\mu(y) \\
&\quad \leq \frac{CR^{\alpha(p-1)}}{\gamma^p} \int_X |f(y)|^p R^\alpha M\chi_{B(a,r)}(y)\, d\mu(y).
\end{aligned}$$

Furthermore, Theorem 2.1 enables us to find

$$\begin{aligned}
\mu\left(\{x \in B(a,r) : |I_\alpha f(x)| > \gamma\}\right) &\leq \frac{CR^{\alpha(p-1)} R^\alpha r^{n\lambda}}{\gamma^p} \|f\|^p_{L^{p,\lambda}(\mu)} \\
&= Cr^{n\lambda}\left(\frac{R^\alpha \|f\|_{L^{p,\lambda}(\mu)}}{\gamma}\right)^p.
\end{aligned}$$

As $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n(1-\lambda)}$ give us

$$\left(\frac{R^\alpha \|f\|_{L^{p,\lambda}(\mu)}}{\gamma}\right)^p = CR^{n(1-\lambda)} = \left(\frac{\|f\|_{L^{p,\lambda}(\mu)}}{\gamma}\right)^q,$$

then we end up with

$$\mu\left(\{x \in B(a,r) : |I_\alpha f(x)| > \gamma\}\right) \leq Cr^{n\lambda}\left(\frac{\|f\|_{L^{p,\lambda}(\mu)}}{\gamma}\right)^q.$$

This is our desired inequality. □

3. Concluding Remarks

The results in this paper can be reduced to the result in (Sihwaningrum, 2016) if $p = 1$. Besides, the proof of the weak-(p, q) inequality for fractional integral operator on Morrey spaces over metric measure spaces of nonhomogeneous type can be found by using other methods. One of the common methods is a proof by using Hedberg type inequality.

Acknowledgments. This paper is a result of Fundamental Research Program 2015, Directorate General of Higher Education, Indonesia, No. 2101/UN23.14/PN/2015.

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