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THE ANALYSIS OF MULTIDIMENSIONAL ANOMALOUS DIFFUSION EQUATION

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ABSTRACT. *We discuss the properties of the fundamental solution of multidimensional anomalous diffusion equation such as symmetric, decay, nonnegative, normality, and bounded in mathematical analysis approach.*

Keywords. *fundamental solution, anomalous diffusion, symmetric, decay, nonnegative, normal.*

ABSTRAK. Makalah ini membahas sifat-sifat dari penyelesaian fundamental dari persamaan difusi anomali seperti simetri, luruh, nonnegatifan, dan normal dengan menggunakan pendekatan matematika analisis.

Kata Kunci. Penyelesaian fundamental, difusi anomali, simetrian, luruh, nonnegatif, normal.

1. INTRODUCTION

Anomalous diffusion is a diffusion process which has the characteristic that the mean square displacement (MSD) of a particle moving in the process with respect to time $t > 0$ follows the pattern

$$\langle x^2(t)\rangle \sim t^\alpha, \;\; \alpha > 0, \alpha \neq 1.$$

When $\alpha = 1$, the process is diffusion. If $0 < \alpha < 1$, the process is called subdiffusion or slow diffusion. This kind of anomalous diffusion is modelled by the equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = D_\alpha \Delta u, \;\; 0 < \alpha < 1,$$

where $u = u(x,t)$ denotes the concentration of the particle at site $x \in \mathbb{R}^n$ and time $t > 0$, D_α is a subdiffusion coefficient, and d^α/dt^α is Caputo fractional time derivative defined by

$$\frac{d^\alpha}{dt^\alpha} f(t) = \int_0^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} \cdot \frac{d}{d\tau} f(\tau) d\tau, \ \ 0 < \alpha < 1.$$

When $\gamma > 1$, the process is called superdiffusion or fast diffusion and modelled by the equation

$$\frac{\partial u}{\partial t} = -C_\beta (-\Delta)^{\frac{\beta}{2}} u, \ \ 0 < \beta < 2,$$

where C_β is a superdiffusion coefficient and $(-\Delta)^{\beta/2}$ is called fractional laplacian operator which has the property

$$\mathcal{F}\big((-\Delta)^{\beta/2} u\big)(\xi) = |\xi|^\beta \mathcal{F}(u)(\xi)$$

where $\mathcal{F}$ denotes the fourier transform operator defined by

$$\mathcal{F}\big(u(s)\big)(\xi) = \int_0^\infty e^{-s\xi} u(s) ds.$$

If $\beta = 2$, the operator is called Laplacian.

Here, we discuss the properties of the fundamental solution of the general form of both equation, that is

$$\frac{\partial^\alpha u}{\partial t^\alpha} = -(-\Delta)^{\frac{\beta}{2}} u, \ \ t > 0,$$

$$u(x, 0) = \delta(x), \ \ x \in \mathbb{R}^n,$$

where $0 < \alpha < 1, 0 < \beta < 2$. The properties of the solution discussed here are symmetric, decay, nonnegative, and normal.

This paper is composed of four sections. In the second section, we explain briefly a special function which is called Mittag-Leffler function. We show our main results in the third section. Finally, in the last section, the conclusion of our discussion is given.

2. PRELIMINARIES

We here introduce the Mittag-Leffler function defined by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \ \ \alpha, \beta > 0, \ z \in \mathbb{C}.$$

This function is entire. For $\beta = 1$, we set $E_{\alpha,\beta}(z) = E_\alpha(z)$, and, for $\alpha = \beta = 1$, we have that $E_{\alpha,\beta}(z)$ is nothing but exponential function e^z.

We next give the asymptotic formulas for the Mittag-Leffler function. For $0 < \alpha < 2$, β is an arbitrary complex number, and μ is an arbitrary number such that

$$\frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\},$$

then, for an arbitrary integer $p \geq 1$, the following hold, those are

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} e^{1/\alpha} - \sum_{n=1}^{p} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + O(|z|^{-1-p}),$$

$$|z| \to \infty, \ \ |\arg(z)| \leq \mu,$$

and

$$E_{\alpha,\beta}(z) = - \sum_{n=1}^{p} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + O(|z|^{-1-p}), \qquad |z| \to \infty, \ \ \mu \leq |\arg(z)| \leq \mu.$$

The following well known Proposition tell us the application of Mittag-Leffler function to fractional ordinary differential equation.

Proposition 1. Let $\lambda \in \mathbb{C}$ and f be given complex function defined in $(0, \infty)$. If $v: [0, \infty) \to \mathbb{C}$ is a continuous function solving the fractional ordinary differential equation

$$\frac{d^\alpha}{dt^\alpha} f(t) = \lambda v(t) + f(t), \ \ t > 0$$

$$v(0) = v_0,$$

then it is given uniquely by

$$v(t) = E_\alpha(\lambda t^\alpha) v_0 + \int_0^t (t - s)^{\alpha - 1} E_{\alpha,\alpha}(\lambda (t - s)^\alpha) f(s) ds\,, \ \ t > 0.$$

For more details concerning the fractional integrals and derivatives and the Mittag-Lefller function, we refer to Podlubny [5].

3. MAIN RESULTS

In this section, we study some properties of the fundamental solutions of anomalous diffusion equation : symmetric, decay, nonnegative, and bounded. Let us consider the initial value problem

$$\frac{\partial^\alpha u}{\partial t^\alpha} + (-\Delta)^{\beta/2} u = 0, \text{ in } \mathbb{R}^n \times (0, \infty)$$

$$u(\cdot\,, 0) = u_0, \text{ in } \mathbb{R}^n,$$

$$u(x,t) = 0, \; t > 0, |x| \to \infty,$$

where $0 < \alpha < 1, 0 < \beta < 2$. We transform the problem into Fourier domain. Then

$$\frac{\partial^\alpha}{\partial t^\alpha} \mathcal{F}(u)(k,t) + |k|^\beta \mathcal{F}(u)(k,t) = 0, \text{ in } (0,\infty)$$

$$\mathcal{F}(u)(k,0) = \mathcal{F}(u_0)(k).$$

Therefore, by Proposition 1, we get

$$\mathcal{F}(u)(k,t) = \mathcal{F}(u_0)(k) E_\alpha\left(-|k|^\beta t^\alpha\right).$$

Then, we obtain the fundamental solution to the problem

$$\begin{aligned} u(x,t) &= \mathcal{F}^{-1}\left(\mathcal{F}(u_0)(k) E_\alpha\left(-|k|^\beta t^\alpha\right)\right) \\ &= \mathcal{F}^{-1}\left(\mathcal{F}(u_0)(k)\right)(x) * \mathcal{F}^{-1}\left(E_\alpha\left(-|k|^\beta t^\alpha\right)\right)(x,t) \\ &= u_0(x) * G_{\alpha,\beta}(x,t) \\ &= \int_{\mathbb{R}^n} u_0(y) G_{\alpha,\beta}(x-y,t)\, dy \end{aligned}$$

where

$$\begin{aligned} G_{\alpha,\beta}(x,t) &= \mathcal{F}^{-1}\left(E_\alpha\left(-|k|^\beta t^\alpha\right)\right)(x,t) \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-ik\cdot x} E_\alpha\left(-|k|^\beta t^\alpha\right) dk \end{aligned}$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-ik\cdot x} E_\alpha \left(-\left| k t^{\alpha/\beta} \right|^\beta \right) dk$$

$$= \frac{t^{-n\alpha/\beta}}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-il\cdot\left(x t^{-\alpha/\beta}\right)} E_\alpha\left(-|l|^\beta\right) dl$$

$$= t^{-n\alpha/\beta} K_{\alpha,\beta}\left(x t^{-\alpha/\beta}\right)$$

and

$$K_{\alpha,\beta}(y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-ik\cdot y} E_\alpha\left(-|k|^\beta\right) dk.$$

When $u_0(x) = \delta(x)$, Dirac Delta function, we have that $u(x,t) = G_{\alpha,\beta}(x,t)$, anomalous "Green function". Henceforth, we assume the case.

Next, since $E_\alpha\left(-|k|^\beta\right)$ is symmetric, $G_{\alpha,\beta}(x,t)$ is also symmetric in space domain. Consider now

$$S_N(k) := \sum_{j=0}^{N} \frac{|k|^{\beta j}}{\Gamma(\alpha j + 1)} \to E_\alpha\left(|k|^\beta\right).$$

We have that $\{S_N(k)\}_0^\infty$ is non-decreasing and integrable function sequence on a ball $B_r(0)$ with the center at the origin and radius $r > 0$. By the Monotone Convergence theorem, $E_\alpha\left(|k|^\beta\right)$ is integrable on $B_r(0)$. Since

$$\left| E_\alpha\left(-|k|^\beta\right) \right| \le E_\alpha\left(|k|^\beta\right),$$

we get that $\left| E_\alpha\left(-|k|^\beta\right) \right|$ is also integrable on $B_r(0)$.

We now suppose

$$I(r) = \int_{B_r(0)} \left| E_\alpha\left(-|k|^\beta\right) \right| dk\,,\ r > 0.$$

From [5], we have, for $0 < \alpha < 1$,

$$E_\alpha(z) = -\sum_{j=1}^{p} \frac{z^{-j}}{\Gamma(1-\alpha j)} + O\left(|z|^{-1-p}\right),\ |z| \to \infty,\ \ \frac{\alpha\pi}{2} < \arg z < 2\pi - \frac{\alpha\pi}{2}.$$

Note that $-|k|^\beta$ is non-positive real number (a complex number with arg $= \pi$). Then, for $0 < \alpha < 1$,

$$\left|E_\alpha\left(-|k|^\beta\right)\right| = \left|-\sum_{j=1}^{p} \frac{(-1)^j |k|^{-\beta j}}{\Gamma(1-\alpha j)}\right| = \sum_{j=1}^{p} \frac{|k|^{-\beta j}}{\Gamma(1-\alpha j)} \to 0, \text{ as } |k| \to \infty.$$

Then, for $r < s$,

$$|I(r) - I(s)| = \int_{B_s(0)\setminus B_r(0)} \left|E_\alpha\left(-|k|^\beta\right)\right| dk \to 0, \text{ as } r \to \infty.$$

It means $I(r)$ is Cauchy sequence, and, hence,

$$I(r) \to \int_{\mathbb{R}^n} \left|E_\alpha\left(-|k|^\beta\right)\right| dk < \infty, \text{ as } r \to \infty.$$

It means $E_\alpha\left(-|k|^\beta\right) \in L^1(\mathbb{R}^n)$. Therefore, we obtain $K_{\alpha,\beta}(x)$ and $G_{\alpha,\beta}(x,t)$, as the inverse of the Fourier transform of $E_\alpha\left(-|k|^\beta\right)$ and $E_\alpha\left(-|k|^\beta t^\alpha\right)$, respectively, exist. We next use the following theorem to show the decay property of the fundamental solution.

Riemann-Lebesgue Theorem. If $f \in L^1(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} e^{-ik\cdot x} f(x) dx \to 0, \text{ as } |x| \to \infty.$$

By the Riemann-Lebesgue theorem, since $E_\alpha\left(-|k|^\beta\right) \in L^1(\mathbb{R}^n)$ and

$$K_{\alpha,\beta}(y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-ik\cdot y} E_\alpha\left(-|k|^\beta\right) dk,$$

we have $\lim_{|x|\to\infty} K_{\alpha,\beta}(x) = 0$. Therefore,

$$\lim_{|x|\to\infty} G_{\alpha,\beta}(x,t) = \lim_{|x|\to\infty} t^{-n\alpha/\beta} K_\beta\left(x t^{-\alpha/\beta}\right) = 0.$$

We next check the nonnegativity and normality of $G_{\alpha,\beta}(x,t)$. In [6], it was showed that, for $0 \leq \alpha \leq 1$, $E_\alpha(-z)$ is a completely monotonic function, that is, for $z \in \mathbb{R},\ z \geq 0$,

$$(-1)^n \frac{d^n}{dx^n} E_\alpha(-x) \geq 0,\ n = 0,1,2,\cdots.$$

Therefore, we have $E_\alpha\left(-|k|^\beta\right) \geq 0$ and, thus, $G_{\alpha,\beta}(x,t) \geq 0$. Furthermore, for $t > 0$,

$$\begin{aligned}
\left\|G_{\alpha,\beta}(\cdot,t)\right\|_{L^1(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} G_{\alpha,\beta}(x,t)dx \\
&= \int_{\mathbb{R}^n} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-ik\cdot x} E_\alpha\left(-|k|^\beta t^\alpha\right) dk\, dx \\
&= \int_{\mathbb{R}^n} \left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-ik\cdot x} dx\right) E_\alpha\left(-|k|^\beta t^\alpha\right) dk \\
&= \int_{\mathbb{R}^n} \delta(k) E_\alpha\left(-|k|^\beta t^\alpha\right) dk \\
&= E_\alpha(0) \\
&= 1.
\end{aligned}$$

4. CONCLUSION

The fundamental solution to the problem

$$\frac{\partial^\alpha}{\partial t^\alpha} u(x,t) = -(-\Delta)^{\frac{\beta}{2}} u(x,t),\ \ x \in \mathbb{R}^n, t > 0,$$

$$u(x,0) = \delta(x),\ \ x \in \mathbb{R}^n,$$

where $0 < \alpha < 1, 0 < \beta < 2$ is

$$u(x,t) = G_{\alpha,\beta}(x,t)$$

where

$$G_{\alpha,\beta}(x,t) = \frac{t^{-n\alpha/\beta}}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-ik\cdot\left(xt^{-\alpha/\beta}\right)} E_\alpha\left(-|k|^\beta\right) dk$$

which has the properties :

(a) symmetric in space : $G_{\alpha,\beta}(x,t) = G_{\alpha,\beta}(-x,t),\ x \in \mathbb{R}^n$;

(b) decay : $\lim_{|x|\to\infty} G_{\alpha,\beta}(x,t) = 0$;

(c) nonnegative : $G_{\alpha,\beta}(x,t) \geq 0,\ \ x \in \mathbb{R}^n, t > 0$;

(d) normal : $\left\|G_{\alpha,\beta}(\cdot,t)\right\|_{L^1(\mathbb{R}^n)} = 1$.

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