THE EXISTENCE AND UNIQUENESS OF THE MILD SOLUTION TO A
NONLINEAR CAUCHY PROBLEM ASSOCIATED WITH A NONLOCAL
REACTION-DIFFUSION SYSTEM

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ABSTRACT. We study the existence and uniqueness of a mild solution to a nonlinear
Cauchy problem associated with a nonlocal reaction diffusion system by employing the
properties of analytic semigroup operator generated by the linear part of the problem
which is sectorial and then applying Banach Fixed Point Theorem to the problem. We
show that the problem has a unique mild solution under a Lipschitz condition on the
nonlinear part of the problem. An example as an application of the result obtained is also
given.

Keywords : existence, uniqueness, mild solution, nonlinear Cauchy problem, Banach
fixed point theorem.

ABSTRAK. Kita mengkaji keujudan dan ketunggalan penyelesaian lemah masalah
Cauchy nonlinier yang berkaitan dengan sistem reaksi difusi nonlokal dengan
menggunakan sifat-sifat operator semigroup yang dibangkitkan oleh operator pada bagian
liniernya yang bersifat sektorial dan kemudian menerapkan Teorema Titik Tetap Banach
pada masalah tersebut. Kita tunjukkan bahwa masalah tersebut memiliki penyelesaian
lemah yang unik atas kondisi Lipschitz untuk operator pada bagian nonliniernya. Satu
contoh sebagai penerapan hasil yang diperoleh juga diberikan.

Kata kunci : keujudan, ketunggalan, penyelesaian lemah, masalah Cauchy nonlinier,
teorema titik tetap Banach.

1. INTRODUCTION

Consider the system

\[
\frac{\partial q_A}{\partial t} = D_A \Delta q_A - k_A \int_{\mathbb{E}(\mathbb{R}) \cap \Omega} q_B \, dy \cdot q_A, \text{ in } \Omega \times (0, T),
\]
\[
\begin{align*}
\frac{\partial q_B}{\partial t} &= D_B \Delta q_B - k_B \int_{B(\overline{R}) \cap \Omega} q_A dy \cdot q_B, \quad \text{in } \Omega \times (0, T), \\
\frac{\partial q_A}{\partial n} &= \frac{\partial q_B}{\partial n} = 0, \quad \text{on } \partial \Omega \times (0, T),
\end{align*}
\]  
\tag{1.1}

with \( \Omega \subset \mathbb{R}^n \) is a bounded domain with \( C^2 \) boundary. The system describes the chemical reaction

\[
A + B \rightarrow C
\]

that occurs if and only if a pair \( A - B \) molecules are in a distance less than \( R \). A molecule of \( A \) reacts with a molecule of \( B \) to produce a molecule of \( C \) (see [1]).

In this paper, we study the existence and uniqueness of the mild solution to a Cauchy problem associated with the problem (1.1)

\[
D_t u(t) = Au(t) + f(u(t)), \quad 0 < t \leq T
\]
\[u(0) = u_0 \]  
\tag{1.2}

where \( X \) is a Banach space, \( D_t \) is derivative of first order with respect to \( t \), \( A : D(A) \subseteq X \rightarrow X \) is a sectorial linear operator, \( u_0 \in X \), and \( f : X \rightarrow X \) is a nonlinear mapping satisfying the lipschitz condition, that is there exists \( K > 0 \) such that

\[
\|f(u) - f(v)\| \leq K\|u - v\|, \quad u, v \in X.
\]

We employ the properties of analytic semigroup generated by the operator \( A \) of the problem (1.2) and then apply Banach Fixed Point Theorem to show the existence and uniqueness of a mild solution to the problem (1.2) under certain conditions.

This paper is composed of four sections. In the second section, we provide briefly some properties of analytic semigroup. In the third section, we prove the existence and uniqueness of the mild solution to the problem (1.2). In the last section, we give an example as an application of the result obtained.
2. PRELIMINARIES

In this section, we provide briefly some results concerning solution operators for the Cauchy problem

\[
D_t u(t) = Au(t) + f(t), \quad t > 0
\]
\[
u(0) = u_0
\]

DEFINITION 2.1. An operator \( A \) is called sectorial if \( A \) satisfies the properties that there are constant \( \theta \in \left( \frac{\pi}{2}, \pi \right) \) and \( M > 0 \) such that \( \rho(A) \supset S_\theta = \{ \lambda \in \mathbb{C} : \lambda \neq 0, |\arg(\lambda)| < \theta \} \),

\[
\|R(\lambda, A)\| \leq \frac{M}{|\lambda|}, \quad \lambda \in S_\theta,
\]

where \( R(\lambda, A) = (\lambda I - A)^{-1} \) and \( \rho(A) = \{ \lambda \in \mathbb{C} | R(\lambda, A) \text{ is bounded} \} \) are the resolvent operator and resolvent set of \( A \), respectively.

DEFINITION 2.2. For \( r > 0 \) and \( \frac{\pi}{2} < \omega < \theta \), \( \Gamma_{r,\omega} : = \{ \lambda \in \mathbb{C} : |\arg(\lambda)| = \omega, |\lambda| \geq r \} \cup \{ \lambda \in \mathbb{C} : |\arg(\lambda)| \leq \omega, |\lambda| = r \}

The linear operator \( A \) generates a solution operator for the problem (2.1), that is

\[
S(t) = \frac{1}{2\pi i} \int_{\Gamma_{r,\omega}} e^{\lambda t} R(\lambda; A) \, d\lambda, \quad t > 0,
\]

where \( \Gamma_{r,\omega} \) is counterclockwise. By Cauchy’s theorem, the integral form (2.2) is independent of \( r > 0 \) and \( \omega \in (\pi/2, \theta) \).

Let \( B(X) \) be the set of all bounded linear operators on \( X \). The properties of the family \( \{S(t)\}_{t>0} \) are given in the following theorems.

THEOREM 2.3. Let \( A \) be a sectorial operator and \( S(t) \) be the operator defined by (2.2). Then the following statements hold.

(i) \( S(t) \in B(X) \) and there exists \( C_1 > 0 \) such that

\[
\|S(t)\| \leq C_1, \quad t > 0.
\]
(ii) \( S(t) \in B\left( X; D(A) \right) \) for \( t > 0 \) and if \( x \in D(A) \) then \( A S(t)x = S(t)Ax. \) Moreover, there exists \( C_2 > 0 \) such that
\[
\| AS(t) \| \leq C_2 t^{-1}, \quad t > 0,
\]
where \( B\left( X; D(A) \right) = \{ T : X \to D(A) | T \text{ is bounded operator} \} \) and \( D(A) \subseteq X. \)

(iii) The function \( t \mapsto S(t) \) belongs to \( C^\infty ((0, \infty); B(H)) \) and it holds that
\[
S^{(n)}(t) = \frac{1}{2\pi i} \int_{\Gamma_{r,\omega}} e^{\tau \lambda} \lambda^n R(\lambda; A) \ d\lambda, n = 1, 2, 3, ...
\]
and there exists \( M_n > 0, n = 1, 2, 3, ... \), such that
\[
\| S^{(n)}(t) \| \leq M_n t^{-n}, \quad t > 0.
\]
Moreover, it has analytic continuation \( S(z) \) to the sector \( S_{\theta - \pi/2} \) and for
\[ z \in S_{\theta - \pi/2}, \eta \in \left( \frac{\pi}{2}, \theta \right), \]
it holds that
\[
S(z) = \frac{1}{2\pi i} \int_{\Gamma_{r,\eta}} e^{\lambda z} R(\lambda; A) \ d\lambda.
\]

(iv) For \( x \in X, \)
\[
S(t)S(s) = S(t + s), \quad s, t > 0.
\]

**THEOREM 2.4.** Let \( A \) be a sectorial operator and
\[
S(t) = \frac{1}{2\pi i} \int_{\Gamma_{r,\omega}} e^{\tau t} R(\lambda; A) \ d\lambda.
\]
Then the following statements hold.

(i) If \( x \in D(A) \) then \( \lim_{t \to 0^+} S(t)x = x. \)

(ii) For every \( x \in D(A) \) and \( t > 0, \)
\[
\int_0^t S(\tau)x \ d\tau \in D(A),
\]
\[
\int_0^t AS(\tau)x \ d\tau = S(t)x - x.
\]

(iii) If \( x, Ax \in D(A) \) then
\[
\lim_{t \to 0^+} \frac{S(t)x - x}{t} = Ax.
\]
For more discussion concerning analytic semigroup operator and its properties, one can refer to [2,3].

3. MAIN RESULTS

In this section, we study the existence and uniqueness of the solution to the Cauchy problem (1.2)
\[ D_t u(t) = Au(t) + f(u(t)), \quad 0 < t \leq T, \]
\[ u(0) = u_0 \]
where A is a sectorial linear operator, \( f: X \to X \), and \( u_0 \in X \). We start defining the mild solution for this problem.

**DEFINITION 3.1.** A function \( u: [0, T] \to X \) is a mild solution to the problem (1.2) if it satisfies
\[ u(t) = S(t)u_0 + \int_0^t S(t-s)f(u(s))ds, \quad 0 < t \leq T. \quad (3.1) \]

The following theorem shows us the existence and uniqueness of the mild solution to the Problem (3.1) by assuming the Lipschitz condition on \( f \).

**THEOREM 3.2.** Let \( f: X \to X \) and there exists \( K > 0 \) such that
\[ \|f(u) - f(v)\| \leq K\|u - v\|, \quad u, v \in X. \]
If \( u_0 \in \overline{D(A)} \) then there exists \( T_0 > 0 \) such that the problem (2.1) has a unique mild solution \( u \in C([0, T_0]; X) \).

**Proof.** Given \( \varepsilon > 0 \). Since \( u_0 \in \overline{D(A)} \), we have \( \|S(t)u_0 - u_0\| \to 0 \) as \( t \to 0^+ \) by Theorem 2.4(i). It means that there exists \( 0 < \tau \leq T \), such that
\[ \|S(t)u_0 - u_0\| \leq \frac{\varepsilon}{2}, \quad 0 < \tau \leq T. \]
We next suppose the Banach space \( Z = C([0, T_0]; X) \) and its subset \( Y = \{u \in Z: \|u - u_0\|_Z \leq \varepsilon\} \) with
\[ T_0 = \min \left\{ \tau, \frac{\varepsilon}{2C(K\varepsilon + \|f(u_0)\|)} \right\}. \]
We now define a mapping $F$ on $Y$ by

\[ F(u_0) := u_0; \quad Fu(t) := S(t)u_0 + \int_0^t S(t-s)f(u(s)) \, ds, \quad 0 < t \leq T_0. \]

We prove that $Fu \in Y$ or, in other words, $Fu \in Z$ and $\| Fu - u_0 \| \leq \varepsilon$. Note that

\[
\| Fu(t) - u_0 \| = \| Fu(t) - S(t)u_0 + S(t)u_0 - u_0 \|
\leq \| Fu(t) - S(t)u_0 \| + \| S(t)u_0 - u_0 \|
= \left\| \int_0^t S(t-s)f(u(s)) \, ds \right\| + \| S(t)u_0 - u_0 \|
\leq \int_0^t \| S(t-s) \| \| f(u(s)) \| \, ds + \| S(t)u_0 - u_0 \|.
\]

By Theorem 2.3(i), we obtain

\[
\| Fu(t) - u_0 \| \leq C \int_0^t \| f(u(s)) \| \, ds + \| S(t)u_0 - u_0 \|.
\]

By using Lipschitz condition, and since $u \in Y$, for $0 < s \leq t \leq T_0 < \tau$, we obtain

\[
\| f(u(s)) \| \leq K \varepsilon + \| f(u_0) \|. \tag{3.2}
\]

Therefore

\[
\| Fu(t) - u_0 \| \leq C (K \varepsilon + \| f(u_0) \|) T_0 + \frac{\varepsilon}{2}.
\]

implying

\[
\| Fu - u_0 \| \leq C (K \varepsilon + \| f(u_0) \|) T_0 + \frac{\varepsilon}{2} < \varepsilon.
\]

We now prove that $Fu \in Z$ or continuity of $Fu$ in $t \in [0, T_0]$. Consider that, for $0 < t \leq T_0$.

\[
\| Fu(t + h) - Fu(t) \|
\leq \| S(t + h)u_0 - S(t)u_0 \|
+ \left\| \int_0^{t+h} S(t + h - s)f(u(s)) \, ds - \int_0^t S(t-s)f(u(s)) \, ds \right\|
\leq \| S(t + h)u_0 - S(t)u_0 \| + \int_0^t \| S(t-s) \| \| f(u(s + h)) - f(u(s)) \| \, ds
+ \int_0^h \| S(t + h - s) f(u(s)) \| \, ds
\]
\[ \leq \|S(h)S(t)u_0 - S(t)u_0\| + \int_0^t \|S(t-s)\| \|f(u(s) + h) - f(u(s))\| \, ds + C(K\varepsilon + \|f(u_0)\|)h \]

By (3.2), the Lipschitz condition on \( f \), and the Dominated Convergence Theorem,
\[ \int_0^t \|S(t-s)\| \|f(u(s) + h) - f(u(s))\| \, ds, \text{ as } h \to 0. \]

Therefore \( \|Fu(t + h) - Fu(t)\| \to 0 \) as \( h \to 0 \) for \( 0 < t \leq T_0 \). For \( t = 0 \), we have
\[ \|Fu(h) - Fu(0)\| = \|Fu(h) - u_0\| \]
\[ \leq \|S(h)u_0 - u_0\| + \int_0^h \|S(h-s)f(u(s))\| \, ds \]
\[ \leq \|S(h)u_0 - u_0\| + C(K\varepsilon + \|f(u_0)\|)h. \]

Then \( \|Fu(t + h) - Fu(t)\| \to 0 \) as \( h \to 0 \) for \( t = 0 \). Consequently, \( Fu \)
 continuous in \( t \in [0,T_0] \) or \( Fu \in Z \). Thus \( F: Y \to Y \).

Next, we prove that \( F: Y \to Y \) is a contraction. Observe that
\[ \|Fu(t) - Fv(t)\| \]
\[ = \left\| S(t)u_0 + \int_0^t S(t-s)f(u(s)) \, ds - S(t)u_0 - \int_0^t S(t-s)f(v(s)) \, ds \right\| \]
\[ \leq \int_0^t \|S(t-s)\| \|f(u(s)) - f(v(s))\| \, ds \]
\[ \leq CK \int_0^t \|u(s) - v(s)\| \, ds. \]

Since \( u \in Y \subseteq Z = \{u: [0,T_0] \to X : u \text{ is continuous on } [0,T_0]\}, \)
\[ \|Fu(t) - Fv(t)\| \leq T_0 CK \|u - v\|_Z. \]

This implies
\[ \|Fu - Fv\|_Z \leq T_0 CK \|u - v\|_Z. \]

Since
\[ T_0 CK\varepsilon < T_0 C \left( K\varepsilon + \|f(u_0)\|_{L^\infty([0,T];X)} \right) \leq \frac{\varepsilon}{2}, \]
we have $T_0CK < 1/2$. Therefore $F: Y \to Y$ is a contraction. By Banach Fixed Point Theorem, $F$ has a unique fixed point $u \in Y$. Here we have

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(u(s))\,ds, \quad 0 < t \leq T.$$ 

Note that since $u_0 \in \overline{D(A)}$, then by 3.2, Theorem 2.3(i), and Theorem 4.12(i), we obtain $\|u(t) - u_0\| \to 0$ as $t \to 0^+$. It follows that $u \in C([0,T_0]; X).

4. APPLICATIONS

Consider again the system (1.1)

$$\frac{\partial q_A}{\partial t} = D_A\Delta q_A - k_A \int_{B(\cdot, R) \cap \Omega} q_B \, dy \cdot q_A, \quad \text{in} \ \Omega \times (0,T),$$

$$\frac{\partial q_B}{\partial t} = D_B\Delta q_B - k_B \int_{B(\cdot, R) \cap \Omega} q_A \, dy \cdot q_B, \quad \text{in} \ \Omega \times (0,T),$$

$$\frac{\partial q_A}{\partial n} = \frac{\partial q_B}{\partial n} = 0, \quad \text{on} \ \partial \Omega \times (0,T),$$

with $\Omega \subset \mathbb{R}^n$ is a bounded domain with $C^2$ boundary. The abstract formulation of the problem (1.1) is

$$D_t U = AU + F(U), \quad 0 < t \leq T,$$

$$U(0) = U_0$$

in

$$X = \{ (u,v) : u, v \in L^2(\Omega) \}$$

where

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, U_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, D_t U = \begin{pmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial t} \end{pmatrix}, A = \begin{bmatrix} D_A\Delta & 0 \\ 0 & D_B\Delta \end{bmatrix}, \text{and} \quad F = \begin{pmatrix} f(u,v) \\ f(v,u) \end{pmatrix}$$

with

$$f(u,v) = uv \text{ and } \tilde{v} = \int_{\Omega} \chi(\cdot, y)v(y)\,dy,$$
where \( \chi(x, y) = \chi_{B(x, R) \cap \Omega}(y) \) denoting the characteristic function of \( B(x, R) \cap \Omega \).

We next set
\[
D(A) = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} : u, v \in H^2_N(\Omega) \right\}
\]
where
\[
H^2_N(\Omega) = \left\{ u \in H^2(\Omega) : \frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega \right\}.
\]
The operator \( \Delta \) is dissipative and self-adjoint implying that \( \Delta \) is sectorial in \( X \).

Moreover, for any \( \lambda \in S_\theta \) with \( \theta \in (\pi/2, \pi) \), we have
\[
(\lambda - A)^{-1} = \begin{bmatrix}
(\lambda - D_A \Delta)^{-1} & 0 \\
0 & (\lambda - D_B \Delta)^{-1}
\end{bmatrix}.
\]
Thus, there exists \( M > 0 \) such that \( \| (\lambda - A)^{-1} \| \leq M/|\lambda| \) for all \( \lambda \in S_\theta \).

Note that since
\[
\| \bar{u} \|_2^2 \leq |\Omega|^4 \| u \|_2^2,
\]
then, for
\[
U = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \quad V = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \in D(A),
\]
we get
\[
\| F(U) - F(V) \|_2^2 \\
\leq \| u_1 \bar{v}_1 - u_2 \bar{v}_2 \|_2^2 + \| v_1 u_1 - v_2 u_2 \|_2^2 \\
= \| (u_1 - u_2) \bar{v}_1 + u_2 (\bar{v}_1 - \bar{v}_2) \|_2^2 + \| (v_1 - v_2) u_1 + v_2 (\bar{u}_1 - \bar{u}_2) \|_2^2 \\
\leq (\| u_1 - u_2 \|_2 \| \bar{v}_1 \|_2 + \| u_2 \|_2 \| \bar{v}_1 - \bar{v}_2 \|_2)^2 \\
+ (\| v_1 - v_2 \|_2 \| \bar{u}_1 \|_2 + \| v_2 \|_2 \| \bar{u}_1 - \bar{u}_2 \|_2)^2 \\
\leq |\Omega|^4 (\| u_1 - u_2 \|_2 \| v_1 \|_2 + \| u_2 \|_2 \| v_1 - v_2 \|_2)^2 \\
+ (\| v_1 - v_2 \|_2 \| u_1 \|_2 + \| v_2 \|_2 \| u_1 - u_2 \|_2)^2 \\
\leq |\Omega|^4 (\| u_1 - u_2 \|_2 + \| v_1 - v_2 \|_2)^2 \\
+ (\| v_1 \|_2 + \| u_2 \|_2)^2 + (\| u_1 \|_2 + \| v_2 \|_2)^2 \\
\leq 4|\Omega|^4 (\| u_1 - u_2 \|_2^2 + \| v_1 - v_2 \|_2^2) \\
+ (\| u_1 \|_2^2 + \| u_2 \|_2^2 + \| v_1 \|_2^2 + \| v_2 \|_2^2) \\
= 4|\Omega|^4 (\| U \|_2^2 + \| V \|_2^2) \| U - V \|_2^2.
\]
Consequently, since \( \| (u, v) \|_\infty \leq \| (u_0, v_0) \|_\infty \), there exists \( K > 0 \) such that
\[ \|F(U) - F(V)\|_2 \leq K\|U - V\|_2. \]

Thus, since \( A \) is a sectorial operator, based on Theorem 3.2, if \( U_0 \in \overline{D(A)} \) then the problem (1.1) has a unique mild solution \( U \in C([0, T_0]; X) \) for some \( T_0 > 0 \).

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